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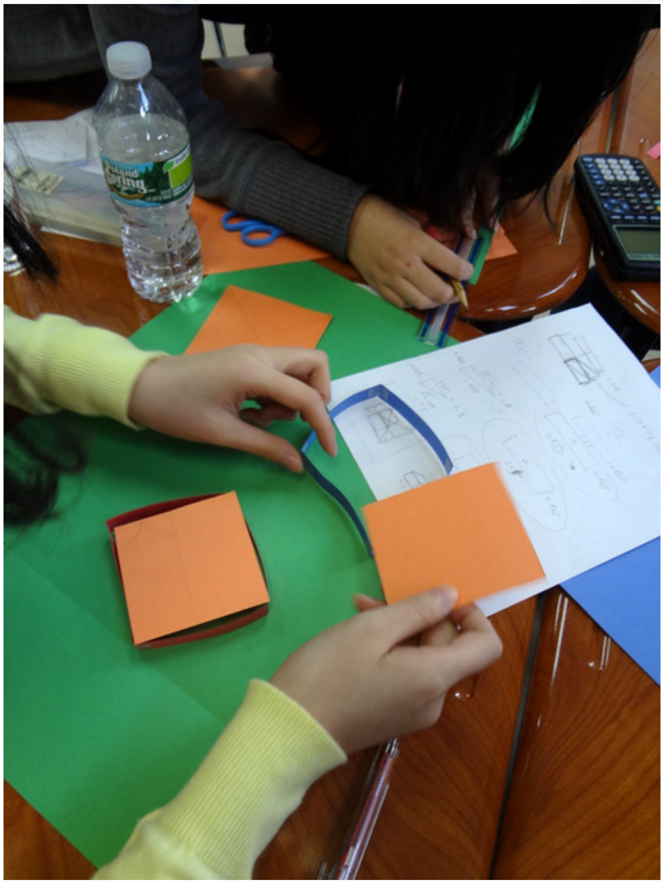
THE 2012 ROSENTHAL PRIZE

for Innovation in Math Teaching

Sphere Dressing



Lesson Plan

GRADES

4-12

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Table of Contents

Note: This lesson plan is based on a submission to the 2012 Rosenthal Prize by runner-up Patrick Honner.

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Overview

In this lesson, students explore the geometry of polygons and their relationships to geometric surfaces and solids by designing and making "hats" to dress a sphere.

The basic directive is a challenge to students to design and construct a hat that meets a certain set of mathematical specifications. A styrofoam sphere serves as the "model" for the students, and ultimately the hats will be displayed and evaluated atop the sphere. The specifications of the activity can be tailored in many ways to address different instructional goals and different grade levels and abilities.

By participating in this activity, students will explore and apply properties of triangles, quadrilaterals, regular polygons, and other shapes. Through physical construction, they will directly experience the relationships between polygons, surfaces, and solids, and they will use geometry to solve real-world problems. In addition, they will engage in mathematical practices like measurement, modeling, using appropriate tools strategically, and attending to precision.

Prior Knowledge Requirements

Students will need to be familiar with properties of regular polygons so that they can construct them. Knowledge of the relationship of polygons and their inscribed circles would be helpful, but the activity could also be structured with this knowledge as a learning objective.

This activity is designed for a range of grades, and it addresses the following standards.

Common Core State Standards

- 6.G Solve real-world and mathematical problems involving area, surface area, and volume.
- 7.G Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.
- 8.G Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.
- G-MG Apply geometric concepts in modeling situations.

Standards for Mathematical Practice

- Make sense of problems and persevere in solving them.
- Model with mathematics.
- Look for and make use of structure.

Specifications of the Activity

Below is the original activity together with several variations that span different grade levels, abilities, and instructional goals. Note that the timing of the lesson is based on a 50-minute total class period. If more or less time is available, or if the lesson is to be delivered over two class periods, adjust the time accordingly.

Original Activity

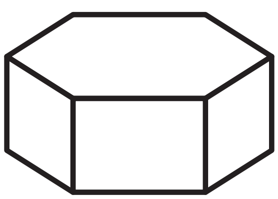
Design and build a hat that has a regular polygon "top" and sides that consist of, at most, two different kinds of polygons. Your hat must cover half the sphere.

Remarks

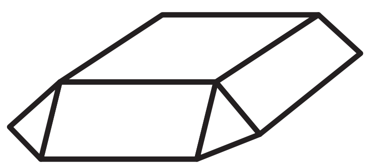
A *regular polygon* is a polygon in which all sides are congruent and all interior angles are congruent. The "top" of the hat is the polygon from which the sides will hang. By *cover half the sphere*, we mean that, when the hat is placed on the sphere with its top horizontal, and the sphere is viewed from the side, exactly half the sphere is visible.

Illustrations

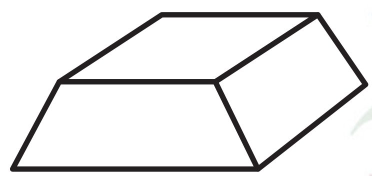
Here are some basic examples of hats that meet the design specifications:



Hexagonal top, rectangular sides



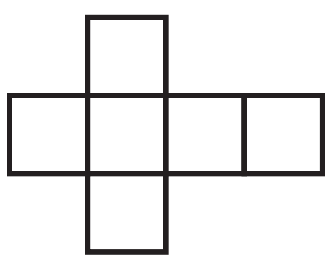
Square top, rectangular and triangular sides



Square top, trapezoidal sides

Terms

A *net* for a three-dimensional shape composed of polygons is a flat layout of polygons connected at edges with lines indicating where it can be folded to create the three-dimensional shape. For example, here is a net for a cube:



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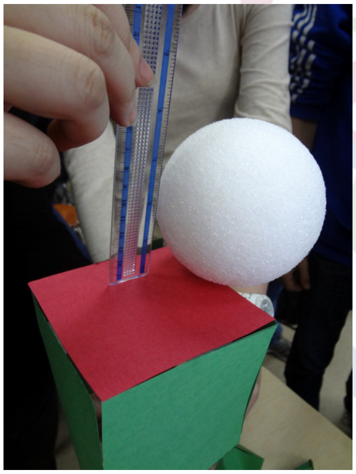
Materials and Preparation



1. A small styrofoam sphere (which can be found in art or fabric supply stores), mounted (e.g., on an unwound coat hanger). One may be placed in the center of the room as the "model", or several may be placed throughout the classroom to expedite measuring and planning.

2. Rulers, tape measures, and protractors for drawing and measuring. String could be useful in measuring the circumference of the sphere. Compasses may be useful, if circles and cones will be part of the activity.

3. Tape, glue, scissors, colored construction paper, and blank paper for designing and constructing the hats. Dot paper (square and triangular) may facilitate making nets.

4. Templates of regular polygons may be appropriate, depending on the age and level of the class and the selected activity.

Description of the Lesson

1. Place the sphere in center of the room, or place several around the room. Inform the students that their task will be to "dress" the sphere, and that at the conclusion of the activity there will be a fashion show. The specific mathematical challenge will be to construct a hat that covers exactly half the sphere. By half the sphere, we mean that, when the hat is placed on the sphere with its top horizontal, exactly half the sphere is visible when viewed from the side.

   The hats must satisfy the following guidelines. Each hat must have a regular polygon as its "top" and can consist of up to two other different kinds of polygonal pieces as "sides". For example, a hat could have a square "top" and trapezoidal "sides"; or it could have a hexagonal "top", and squares and isosceles triangles as "sides".

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Description of the Lesson (Continued)

2. Show the students a few sample hats to give them a sense of what is required and what can be accomplished. These examples could be teacher-created, or created by other classes or groups. Distribute constructions and measurement materials to the groups.

3. Inform the students that the first part of the challenge is measurement and design, and will last 20 minutes. Have each group send a student up to the sphere to take measurements, say by using a ruler, a piece of string, or perhaps a custom measurement device created from project materials. Students might measure the “height” of the sphere (using a ruler), or the circumference of the sphere using string. Have students agree in advance on what system of measurement they will use.

4. Have teams work together to produce a design document that includes diagrams, measurement, and calculations. Require that teams have their designs approved by an instructor before construction begins. The instructor should verify that the teams’ designs meet the given specifications (in this case, that the hat has a regular polygonal top and uses at most two different polygonal shapes for the sides), and that the measurement and calculations are consistent with the plan.

5. The next stage is to build the hats, for which students will have 15 minutes. To produce their hats, students measure, draw, and cut individual polygons from construction paper and tape them together to produce their hats. (If glue is preferred, have students add small “flaps” to each edge of their polygons to provide a surface for glue.) In addition, students may consider using a net to plan and manufacture their hat.

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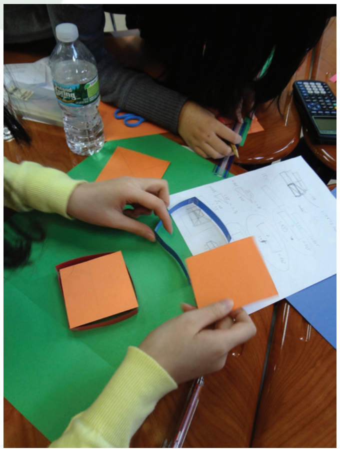
Description of the Lesson (Continued)



6. During the construction phase, circulate among the groups to assess progress and precision. Ask students questions that probe their knowledge of the shapes they are using, the relationships among their shapes, and why they believe their hat will cover half the sphere. Some sample questions are: “What are the angle measurements in this regular polygon?”; “How do you know that this shape is really a regular hexagon?”; “How do you know that all of these rectangles are congruent?”; “Why are you using isosceles triangles here?”; “Can you figure out at which points your hat will touch the sphere?” and, “How tall will your hat be when it is finished, and how do you know?”

   Be prepared to intervene when students are imprecise in drawing and cutting their polygons. Some students may try to “eyeball”, say, a regular hexagon; take this opportunity to make the student explain their figure and defend their choices. Allow students to return to the sphere to take further measurements, but don’t let them “test” their hat until the end.

7. The timing above should leave ten minutes at the end of a 50 minute period for the “fashion show” and discussion. As each hat is placed on the sphere, have the group describe their design and explain why it meets the given specifications. Have teams summarize their design process and discuss the mathematics they used in constructing their hats. Also have them discuss any challenges they faced.

8. Have the class approximately measure how much of the sphere is covered. Have the students decide which hat, or hats, best met the challenge, and invite students to nominate teams for “Most Creative Design”, “Most Colorful Hat”, “Best Looking Hat”, or other such categories.

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Description of the Lesson (Continued)

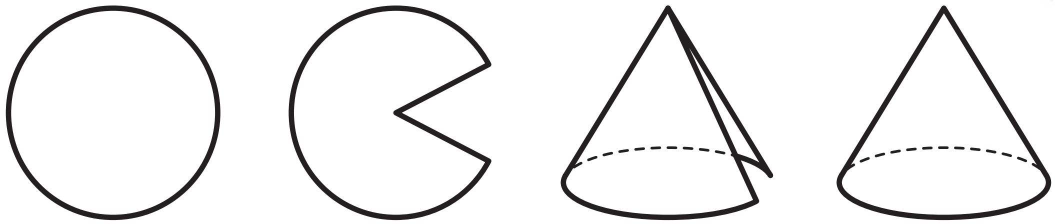
Some Implementation Details

This is a good team-building activity, and works well with groups of four. Suggested roles are *Team Leader & Liaison* (the person in the group responsible for communicating with the instructor), *Measurer* (in charge of taking measurements), *Calculator* (in charge of determining lengths of sides and angles for the needed shapes), *Designer* (in charge of producing the net of the design or its component pieces). Naturally, as with all good group work, every member should participate in all facets of the project.

The activities can work with smaller groups, but one goal of the activity is to foster the exchange of creative, mathematical ideas among students. Smaller groups may reduce that interaction; naturally, the right critical mass depends upon the students and the classroom.

Extensions

The initial constraints help focus the design process, but once students are successful, encourage them to be more creative in designing other hats. Cylinders and cones can be incorporated into the process, as well as irregular shapes and surfaces. For example, a fun game for all grade levels would be creating conical hats by removing a sector of a circle. Have teams experiment with circles and sectors of different sizes in an attempt to cover exactly half the sphere.



These activities would serve well as motivational, or summative, exercises in units on surface area and volume of solid geometric figures. By changing the primary objective from "cover half the sphere" to "enclose the sphere", students create geometric solids whose properties can be explored. Students can research the Platonic solids (*http://www.mathsisfun.com/platonic_solids.html*) and Archimedean solids (*http://en.wikipedia.org/wiki/Archimedean_solid*) to see what kinds of surfaces and solids can be made, and they can explore the properties of those surfaces and solids through measurement and by using nets.

Description of the Lesson (Continued)

By changing the primary objective to "cover half the sphere using the least amount of paper", an optimization problem is created. This activity emphasizes surface area calculations, connects to volume of geometric solids and to real-world problems, and, like the other activities, can be modified to suit various grade and ability levels.

More advanced activities in this vein venture into using curved faces, like **zipper-gons** (*http://bridgesmathart.org/2011/cdrom/proceedings/112/paper_112.pdf*), to efficiently cover the sphere, creating an opportunity to explore advanced ideas like curvature and the Euler characteristic.

Variations on the Basic Activity

The following variations span different grade levels and can easily be modified to fit the learning objectives for classes at different levels of mathematical maturity. Students should be familiar with basic properties of shapes, such as lengths and angles, and should have a basic understanding of the idea of congruence in geometric figures.

Variation 1 (suitable for grades 3-4)

Pass out sets of pre-cut construction paper polygons to groups of students. For example, a set could include a hexagon and six rectangles, or a square and eight equilateral triangles. Have students categorize and classify the polygons based on their shapes and their relationships, such as having congruent sides. Ask students to determine which pieces "fit" together, and challenge them to make a "hat" for the sphere by taping some, or all, of their polygons together.

Variation 2 (suitable for grades 5-6)

Provide the students with sample nets, which are plane diagrams of geometric surfaces or solids (see below). Work with students to fold their nets into surfaces, and ask them to speculate what each surface will look like when the net is folded and assembled. Show the different assembled surfaces as "hats" on the sphere. Then show the students a novel hat, and ask them to draw a net of that hat. Once familiar with nets, and the process of turning polygonal nets into "hats", students can then be tasked with designing their own hat using a net.

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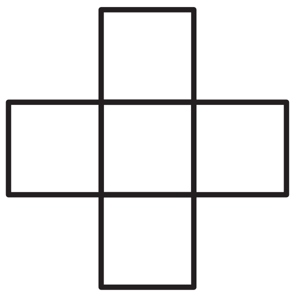
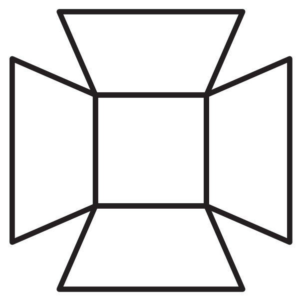
Variations on the Basic Activity (Continued)

Illustrations

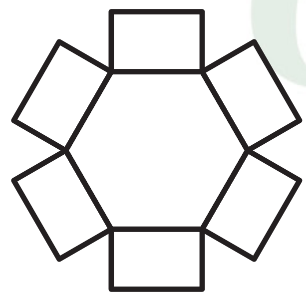
Here are some basic nets. (You can find an overview of nets, and more examples, here: *http://gwydir.demon.co.uk/jo/solid/*)



Open cube



Open trapezoidal frustum



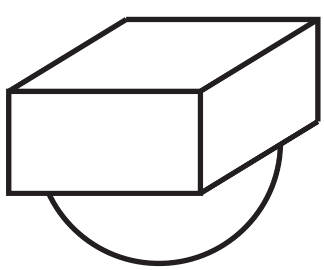
Open hexagonal prism

Variation 3 (suitable for grades 7-8)

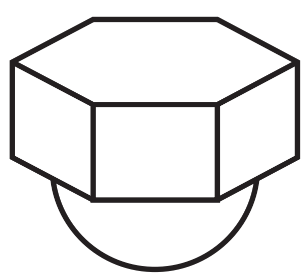
Use the directive of the original activity—design a hat that covers half the sphere—but restrict the design to *open right prisms* with regular polygonal tops. Thus, the hat itself will consist of a regular polygon top and rectangular sides only.

The challenge for students here is to produce a regular polygon top that perfectly "fits" the sphere. "Fit" could be explained as a requirement that the sphere must touch (be tangent to) each side of the polygonal hat. Thus, they will have to find the diameter of the sphere through measurement, and construct a regular polygonal top that "fits" the sphere (i.e., a polygon that circumscribes a great circle of the sphere).

Illustrations



Open rectangular prism



Open hexagonal prism

Variations on the Basic Activity (Continued)

Variation 4 (suitable for high school)

Watch Erik Demaine's *Math Encounters* presentation at MoMath, "The Geometry of Origami" (*http://goo.gl/95qkw*). Then, an advanced challenge would be to ask groups to find the measurements of an aluminum foil rectangle of least area that can wrap the sphere so it is entirely covered. Students may look into Demaine's research to find the Austrian chocolatier whose spheres inspired his work, or compare their solutions to the wrapping of actual spherical candies.

Adapting the Activity to Meet Particular Student Needs

There are a number of ways to adapt the lesson and the various activities to meet the needs of students at different levels of mathematical maturity.

Provide Templates

If students struggle constructing regular polygons from measurements, provide templates of basic shapes, such as squares, triangles, and trapezoids, whose dimensions are known to fit the sphere. Use student success with the template to encourage more ambitious and creative student-generated designs.

Restrict the Shapes Allowed

If students struggle working with too many different kinds of polygons, consider restricting allowable shapes to, say, a square top and trapezoidal sides.

Focus on a Specific Kind of Hat

As suggested in Variation 2 above, consider restricting the hat designs to right prisms, or frustums of pyramids.