The 2024 ROSENTHAL PRIZE
for Innovation and Inspiration
in Math Teaching

Climbing Stairs

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Table of Contents

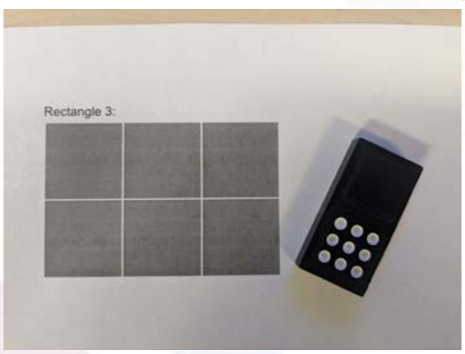
Overview

The lesson “Climbing Stairs” is built around two problems. The following pages have lots of detail about how to pose the problems, how to break them into pieces, how to scaffold them so more students can access the math, and more. All that aside, the heart of the lesson is exploring these two problems with students:

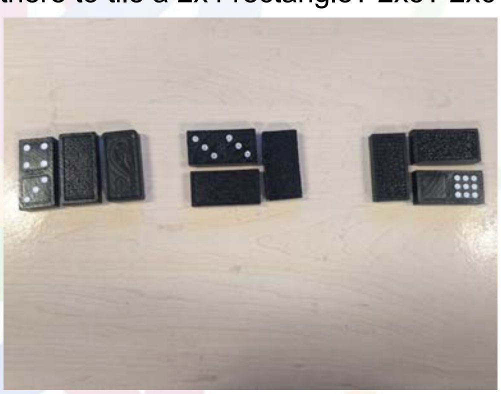
Problem one: Imagine climbing a staircase either 1 or 2 steps at a time. For instance, there are 3 ways to climb a staircase with 3 stairs, pictured below:

Following the same rule, climbing only 1 or 2 steps at a time, how many ways are there to climb a staircase of 4 stairs? 5 stairs? 6? What is the general pattern, and why?

Problem two: Here is a 2x3 rectangle, where a domino is a 1x2 rectangle.



Pictured below are 3 different ways to tile a 2x3 rectangle with dominoes. How many ways are there to tile a 2x4 rectangle? 2x5? 2x6? What is the general pattern, and why?



Goals

The lesson has two goals. The first goal is for students to learn a bit about the Fibonacci sequence, also known as the Hemachandra sequence. It's a fantastic bit of math, accessible to young children with a bit of addition knowledge yet full of surprises. The second goal is for as many students as possible to experience mathematical discovery. That's hard to do! Often it's the same few kids figuring things out. The design of this lesson is to create as many opportunities as possible for mathematical discovery. The **scaffolding** notes in the lesson plan emphasize strategies for helping students access the math in the lesson.

Materials

- A staircase. As few as four stairs is enough, and the ideal staircase is wide enough for a number of students to be on at once. There is a picture of a good example on the following page.
- Day one handout and day two handout, included at the end of this document. Day two handout will need to be adapted to the size of the dominoes you have available. It isn't too hard to create your own hand-drawn handout for this if your dominoes are a different size.
- Dominoes. Enough for 6 per student is enough, though more is great.

Standards

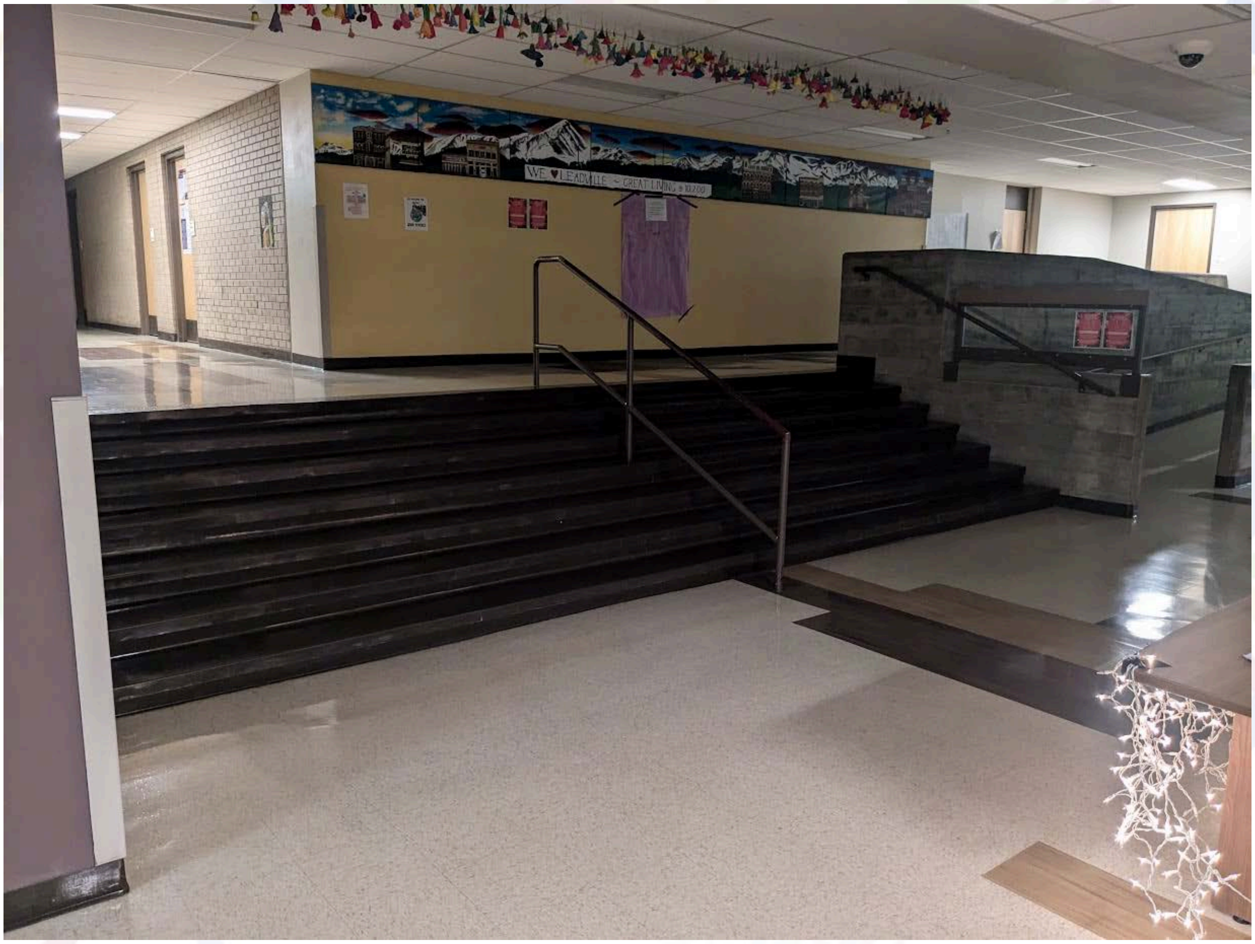
This lesson focuses on Standards for Mathematical Practice:

- SMP.1 Make sense of problems and persevere in solving them
- SMP.3 Construct viable arguments and critique the reasoning of others
- SMP.7 Look for and make use of structure
- SMP.8 Look for and express regularity in repeated reasoning

It's a shame that the Fibonacci sequence does not appear anywhere in the Common Core standards until high school. The sequence is beautiful and accessible to middle school students. While it does not appear explicitly, the sequence connects with the following middle school content standards:
5.OA.3 focuses on analyzing patterns and relationships, of which the Fibonacci sequence is one example
7.RP.2 focuses on proportional relationships, but part of that standard involves discerning whether a relationship is proportional; the Fibonacci sequence is helpful to give students a richer understanding of all of the relationships out there that aren't proportional
The **8.F** strand focuses on functions, of which the Fibonacci sequence is again an example.

Stairs

Use whatever you have available, but a short, wide staircase like this one is ideal:



Time Needed

This lesson plan is written for two periods of about 60 minutes, though the time will vary depending on transitions, time spent on the staircase, and how much work time students need. The first day could easily stretch to 90 minutes depending on your students and context. It is also possible to teach the first day as a standalone lesson and skip the second day.

Lesson Plan

Day One

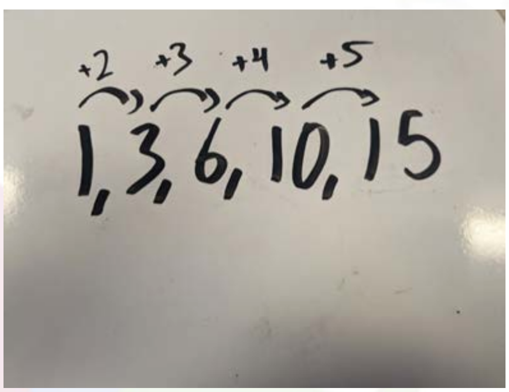
Warmup

The warmup is optional, though highly recommended as a way to get students thinking about patterns before getting to the heart of the lesson. Below are four sequences. For each sequence, ask students: what patterns do they see? A think-pair-share is a good facilitation strategy. This is an accessible way to build a bit of confidence at the start of the lesson, and to prime students to look for different types of patterns in sequences.

Sequences:

- 1, 3, 5, 7, 9, …
- 2, 5, 8, 11, 14, …
- 1, 3, 6, 10, 15, …
- 1, 2, 4, 8, 16, …

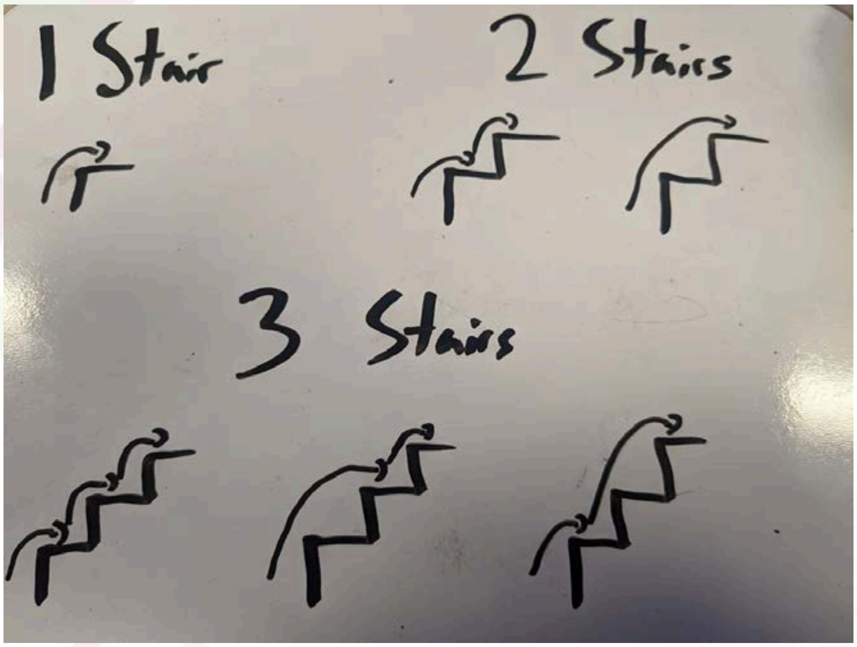
Scaffolding: Keep an eye out for students who notice common differences, for instance noticing that in the second sequence, differences between numbers are +2, +3, +4, etc. This is a helpful strategy for the Fibonacci sequence, and priming it here will help more students make that realization independently later.



Stairs Intro

How many ways are there to walk up a set of stairs, only climbing 1 or 2 stairs at a time? It is important to emphasize that these are the only motions allowed: no steps of 3 or more. You can choose to introduce the problem in a way that feels fun and engaging to you. One option is to take the class to a set of stairs (an example of a good staircase is described above). Ask students to watch you walk up the stairs a few times, and to think about what they notice and what they wonder. Walk up a few times, each time with a different combination of single steps and double steps.

That can lead to some fun questions and exploration. It can also lead to students jumping up and down the stairs and having a grand old time but not doing much mathematical thinking. Next, bring students together to clarify the question. I'm old, so when I climb the stairs I can go up either 1 or 2 stairs with each step. No other steps are allowed: only 1 stair or 2 stairs at a time. The goal of this lesson is to practice looking for patterns, so we'll start small. How many ways are there to go up 1 stair? (Show students the 1 way.) How many ways are there to go up 2 stairs? (Show students the 2 ways.) Who has a prediction for how many ways there are to go up 3 stairs? Ok let's test it. Can you find 3 different ways to climb 3 steps? How are you sure there aren't 4 ways?



Scaffolding: One helpful piece of language to model for students is using "1" and "2" to describe climbing one and two steps, respectively. For instance, for 3 steps the solutions are "1, 1, 1", "2, 1", and "1, 2." This language can help students distinguish between different solutions, and also sets up a connection with the domino problem on day two.

Stairs Work Time Part One

Next, return to the classroom. Put the first three steps of the sequence on the board and frame the next question: "Here are the first three steps of this sequence. I heard some of you make the prediction that there will be 4 ways to climb 4 stairs. Now it's your job to prove it. Can you find 4 ways to climb 4 stairs? How can you be sure there aren't any more?"

Pass out the handout and give students some work time. If students finish with 4 stairs, have them try 5. If they finish 5, have them look for patterns in the numbers they found so far or look at 6 stairs. There are also a few extension questions at the bottom of this section for students who work much faster than others and can benefit from a challenge.

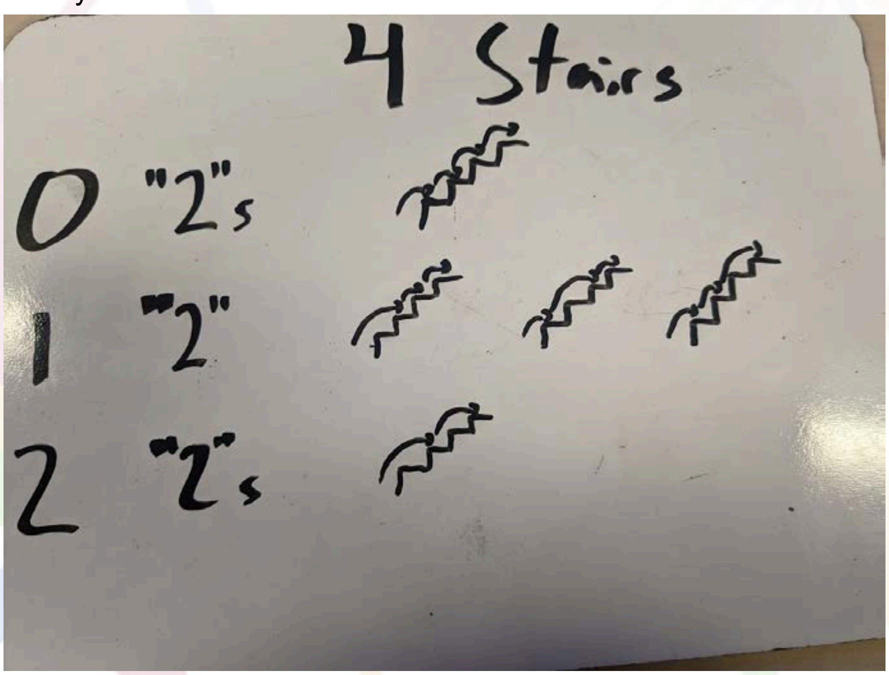
The first goal of this work time is to find solutions for 4 stairs, 5 stairs, and possibly 6 stairs. The second goal is for students to find strategies to organize their work. To be confident that they have found every possible solution, students will need to be systematic in their work. Here are a few other things to watch out for:

- Students may be overconfident that there are 4 ways to climb 4 stairs. There are actually 5! One way to facilitate this is to ask several students to share their work on the board, choosing students so that even if all students only found 4 ways to climb 4 stairs, they have 5 different solutions between them.
- Take your time while going through the different possibilities for 4 stairs. Some students may think they found more than 5 solutions, and it's helpful to work through the answers systematically so all students are convinced there are 5.

When students are ready, facilitate a short debrief. The focus of this debrief will depend on how far students get. Below are several recommendations for the debrief:

- Have a student or students share the solutions they found for 4 stairs. Some students may be very confident that there are only 4 solutions, rather than 5. Others may believe they found more than 5 solutions but have duplicates.
- Have a student or students share the solutions they found for 5 stairs.
- Have a student or students share how they organized their work to be confident they found every solution for 4 stairs or 5 stairs. An example of organized work is at the bottom of the page.
- Discuss how to approach organizing solutions for larger numbers to ensure no solutions are missed.

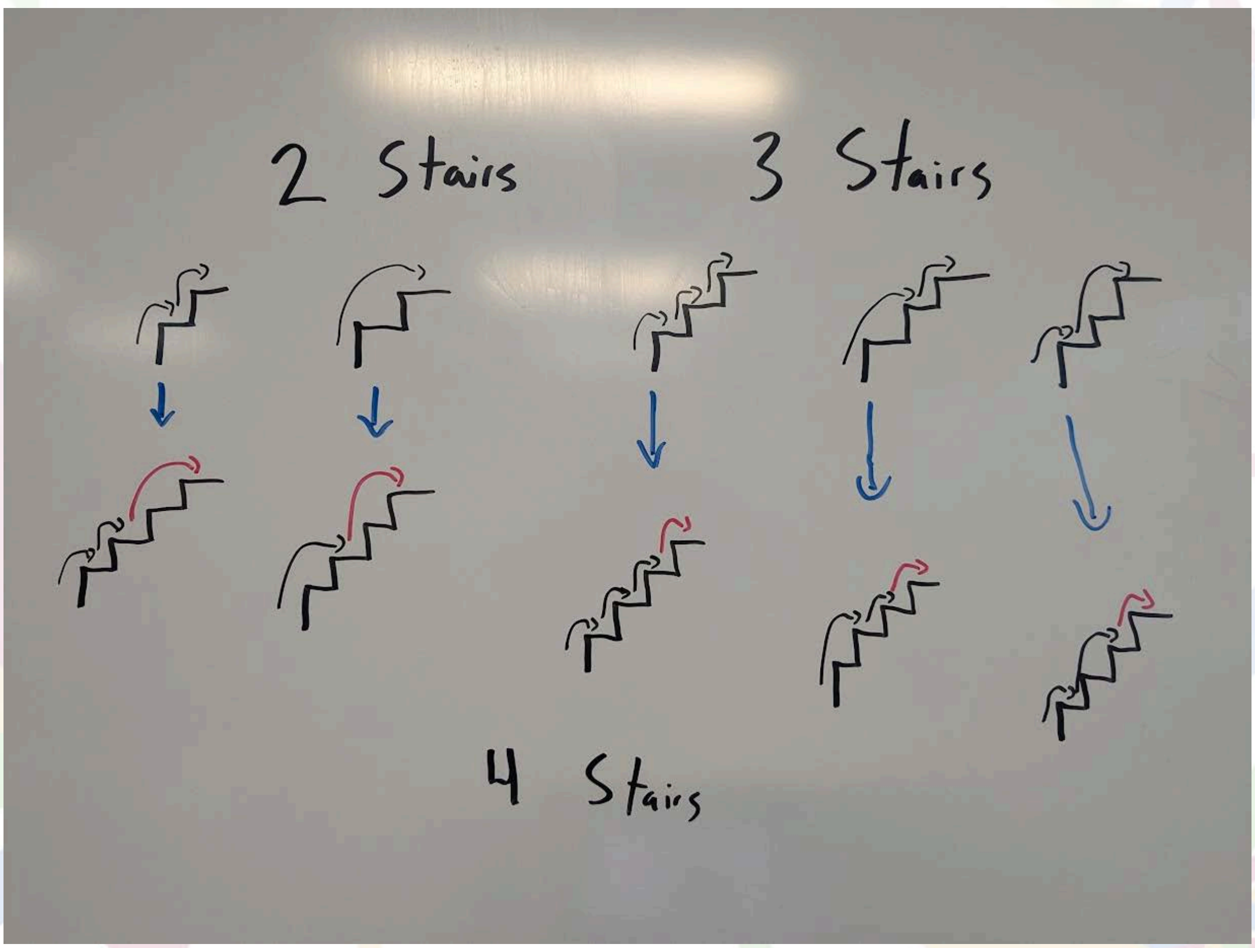
Scaffolding: Returning to the language of “1” and “2” can be helpful for students who struggle to distinguish between answers. Drawings can be ambiguous, but if both drawings refer to “1, 1, 2” they must refer to the same solution.



Stairs Work Time Part Two

What students work on here will depend on how the first round of work goes. Here are a few questions to explore next.

- How many ways are there to climb 5 stairs? 6 stairs? How are students sure they have found every possible way? Solving 6 stairs will be easier after an example of organized work.
- It is becoming harder and harder to count all the ways to climb each number of stairs. What patterns do students see in the numbers? How can they use those patterns to predict what comes next without writing out every possibility?
- Why is this the Fibonacci sequence? There are many other sequences that include 1, 2, 3, 5, 8. Why does this specific problem lead to a sequence where the solutions for 2 stairs added to the solutions for 3 stairs equal the solutions for 4 stairs? Below is a diagram that shows one way to understand this connection. The diagram shows how the solutions for 2 stairs + a 2 step, and the solutions for 3 stairs + a 1 step, can be used to build the solutions for 4 stairs, showing why this sequence is additive.
- At the end of today's section are a few extension questions that students can explore if they have extra time. These are good questions to engage students if they have answered the questions above but you want other students to have more work time to reach their own conclusions.



Scaffolding: These questions are challenging! Slowing students down and focusing on neat, organized, strategically drawn diagrams can be a good strategy to help students notice patterns and make connections.

Closing

To close, give students another opportunity to share what they have discovered and to discuss the strategies they have used. The exact structure of the discussion will depend on your students, but below are a few suggestions:

- How many solutions are there for 5 stairs? 6 stairs? How are you sure?
- What is the pattern? What numbers come next? How are you sure?
- Why is each step of the pattern equal to the two previous steps added together?

After discussing the sequence, this is a good time to close by sharing with students that this sequence has a name, most often called the Fibonacci sequence after a mathematician named Fibonacci, though it was also discovered at a few other points in history, for instance by Hemachandra and others. Though it seems simple, it comes up in surprising places. If you have more knowledge about the sequence, here is a good place to share it.

Scaffolding: If students struggle to see the pattern in the numbers 1, 2, 3, 5, 8, it might be helpful to reference some of the strategies that students used in the warmup problems at the beginning of the lesson to give students a place to start.

Extension Questions

Below are some extension questions to have ready for students who are finished early. These extension questions are a good way to give all students plenty of “think time,” while providing a challenge for students who are ready for it.

- What sequence would you get if you could climb 1, 2, or 3 stairs at a time, rather than just one or two?
- Which Fibonacci numbers are even? Which are odd? Why?
- Which Fibonacci numbers are multiples of three? Why?
- What would happen if you extended the Fibonacci sequence backwards? (Meaning, what comes before 1, 2, 3?)
- What are some other problems that result in the Fibonacci sequence? For instance, the number of ways of making 5¢, 10¢, 15¢, 20¢, etc by using only nickels and dimes results in the same sequence. What other problems can you find with the same answer?

Day Two

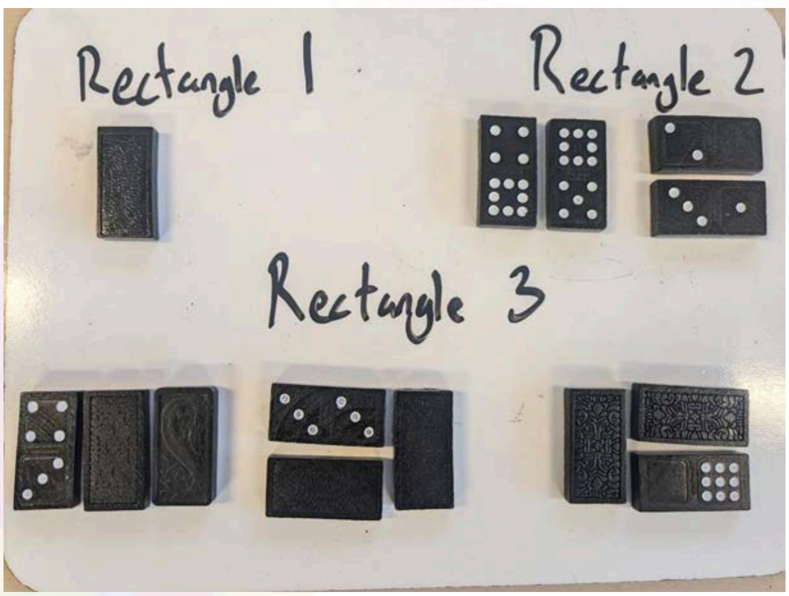
Dominoes Intro

Day two is a bit different than day one. It's also optional – day one is a great lesson on its own, but there is still more to explore if you are game! There is one large problem to work on, and the work time will probably feel similar to day one. The answer (spoiler!) is also the Fibonacci sequence. It's a great problem, and it's worth spending time playing around with. The second day can have a bit of a different flavor. Once it's clear the answer is the Fibonacci sequence, students can focus on trying to understand why two seemingly different problems give the same answer, rather than finding solutions for more and more steps. There are lots of different ways to understand that idea, and it's worth spending time on that part of the discussion.

The basic problem is simple. Provide students with a few 2 x n rectangles, where a domino is a 2 x 1 rectangle. How many different ways are there to tile the large rectangles with dominoes? See the handout at the end of this document for an example of what the rectangles look like. Depending on the size of the dominoes you have access to, it might be easier to draw the rectangles by hand and photocopy your own handout.

This problem does not come from any sort of real-world situation. Instead, frame it as an interesting puzzle to explore and another chance to practice looking for patterns.

Here is an example of what the solutions look like for the first three rectangles:



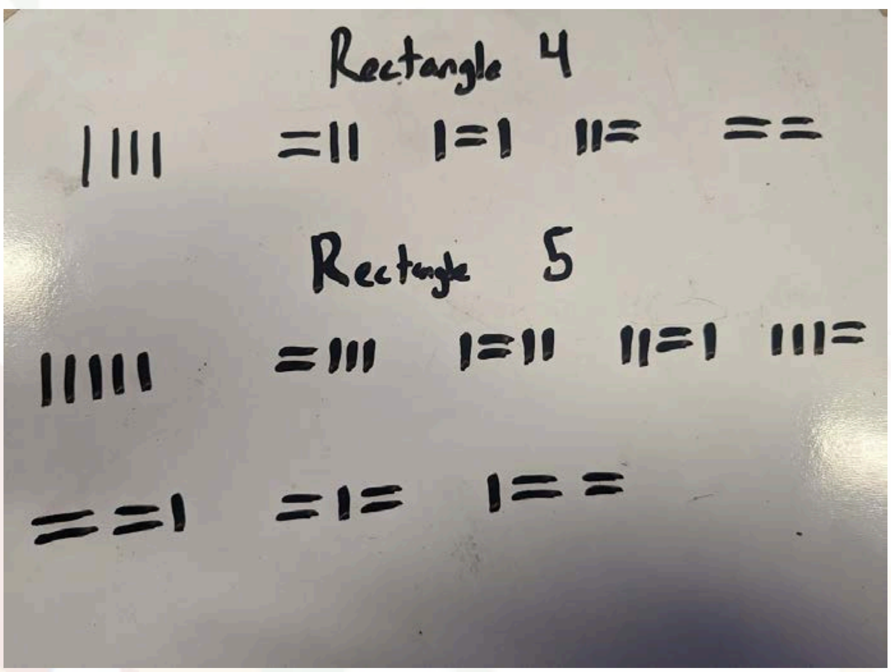
Scaffolding:
There is not enough room on the handout for all of the domino arrangements, so it is helpful to model for students how to organize their work. Students may also get confused by the dots or whether the dominoes are facing up or down. These details don't matter, and carefully modeling this will help students focus on the key mathematical ideas.

Dominoes Work Time Part One

After introducing the problem, give students some work time to try and find the number of ways to arrange the dominoes for the 4th, 5th, and 6th steps. After students have some solutions for those, have students share their answers. The structure will depend on how far students get. A few areas to focus on are listed below:

- Some students may realize quickly that this problem again leads to the Fibonacci sequence. Try to keep that observation from spreading for as long as possible so more students can discover it for themselves.
- Similar to the last problem, after the first round of work time it's helpful to have students who have carefully organized their work to share. This is helpful for students to feel confident that the numbers are accurate, in particular if they think they found more solutions but actually have a duplicate.
- Students are more likely to finish early on this day, which means it's helpful to have a few extension questions ready. The extensions from the Day One lesson are still useful for day two.

Scaffolding: It's helpful to model notation for students to keep track of solutions. Solving all of the problems with dominoes, it is easy to run out of room or get disorganized. Another way to write out the solutions is to use lines to represent dominoes, shown below;



Dominoes Work Time Part Two

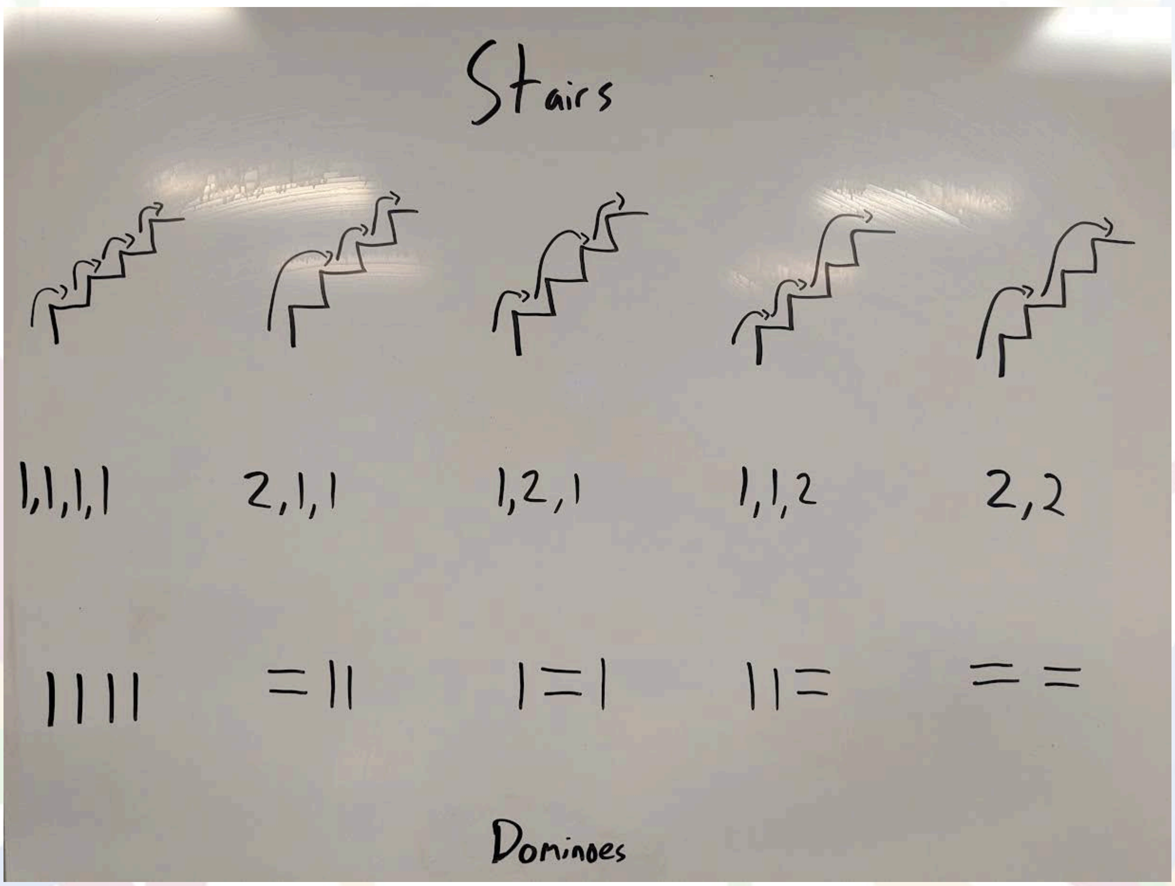
After the first round of work time and debrief, students should feel confident with the answers to the first 5 steps of the sequence and have noted that the sequence is again the Fibonacci sequence. The next questions will step back and think about some of the connections they are seeing:

- Why does this specific problem lead to the Fibonacci sequence? For instance, why do the number of solutions to a 2 x 2 rectangle added to the number of solutions to a 2 x 3 rectangle equal the number of solutions to a 2 x 4 rectangle?
- Why do these two problems that seem very different lead to the same sequence? What do they have in common?

Scaffolding: To set students up for the connection between the stairs problem and the dominoes problem, it's helpful to have concise language to describe different patterns. Note that for every solution, dominoes are either oriented vertically (1) or in a pair oriented horizontally (2). The solutions for rectangle 3 can be described as "1, 1, 1", "2, 1", and "1, 2". This can help students to organize their work and also to connect the dominoes problem to the stairs problem.

Closing

At this point, students likely recognize that the dominoes sequence is the same as the stairs sequence from the previous lesson. This leads to one final question. Why do these two problems that seem very different lead to the same pattern? It's helpful to give students plenty of think time for this question. There are lots of possible answers. It's a ton of fun to talk about with students, and you might learn some new math along the way! Below is one example of how the solutions to the stairs and dominoes problems connect, building off of the 1 and 2 language.



References and Further Reading

I first learned about the stairs problem from Play With Your Math, linked here: https://playwithyourmath.com/2017/07/27/7-step-up/

I first learned about the dominoes problem at the Park City Mathematics Institute, in this problem set: https://projects.ias.edu/pcmi/hstp/resources/course2017.html

A more detailed explanation of why the stairs problem leads to the Fibonacci sequence can be found at Stack Overflow here: https://stackoverflow.com/questions/56698831/n-stair-step-climbing-problem-cannot-conceptualize-why-tn-tn-1-tn-2

The Wikipedia article for the Fibonacci sequence is a good tour of its history and highlights a number of problems in which the sequence arises, including the domino and stairs problems from this lesson: https://en.wikipedia.org/wiki/Fibonacci_sequence

You can also learn more about the Fibonacci sequence in The Online Encyclopedia of Integer Sequences: https://oeis.org/A000045

The Fibonacci sequence is often connected to spirals in the natural world. While some spirals do connect to the Fibonacci sequence, the connection is often stretched to include any spiral in a way that is not accurate. There is a chapter in the book *A Mathematical Nature Walk* by John Adam which has a good, concise summary of why the sequence shows up in nature: https://press.princeton.edu/books/paperback/9780691152653/a-mathematical-nature-walk

The Fibonacci sequence is a common problem in computer science. Coding could be a good extension for some students. More information about a programming approach is here: https://www.geeksforgeeks.org/program-for-nth-fibonacci-number/ and an example built around the stairs problem is here: https://www.geeksforgeeks.org/count-ways-reach-nth-stair/

Handouts

The handouts follow on the next four pages. The handouts are meant to be photocopied two-sided and handed out one at a time. On the day one handout (pages 15-16) there is lots of space for students to find solutions to the 4, 5, and 6 stair problems. To allow room for these steps there is no space for 1, 2, and 3 stairs, so it is important to model these on a whiteboard or other common space. On the day two handout (pages 17-18) there is one rectangle for each step, so students will need to write down their solutions or organize their work another way.

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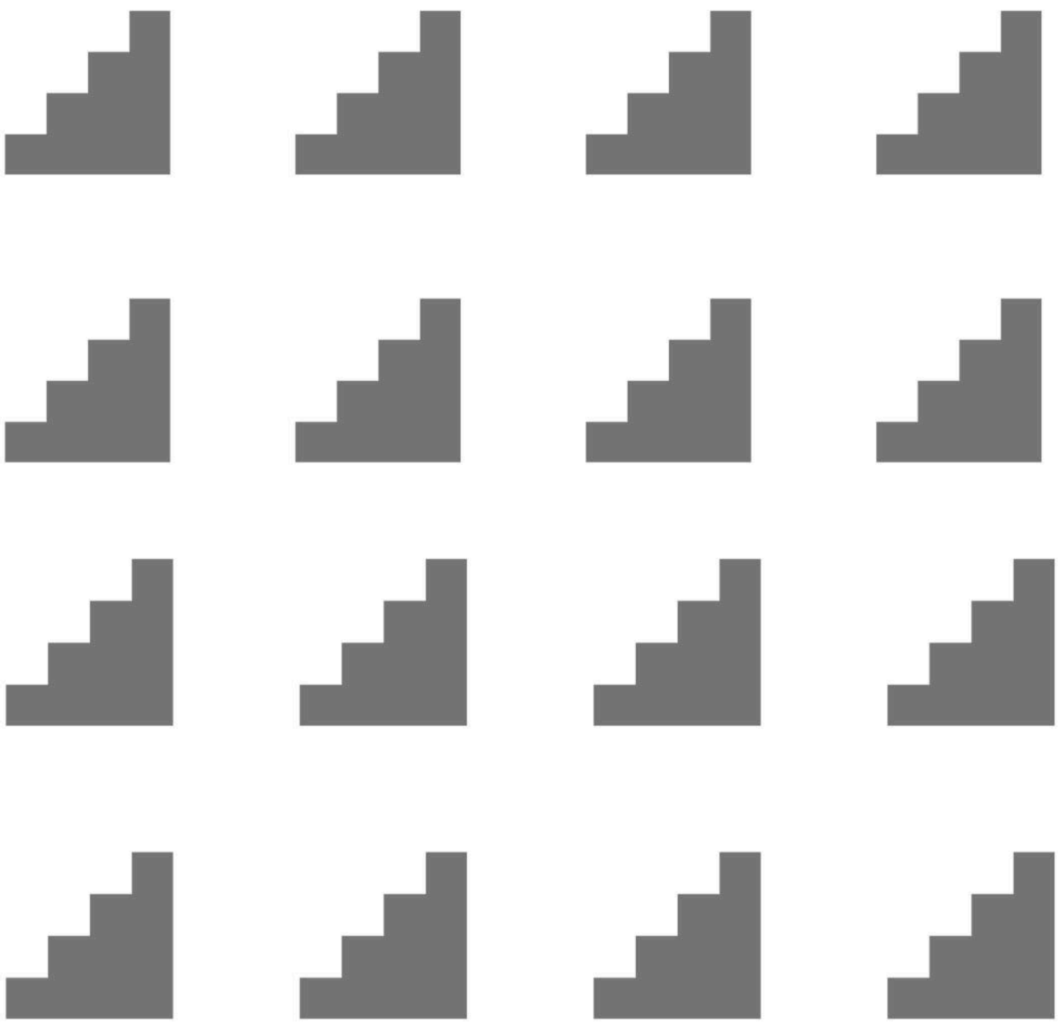
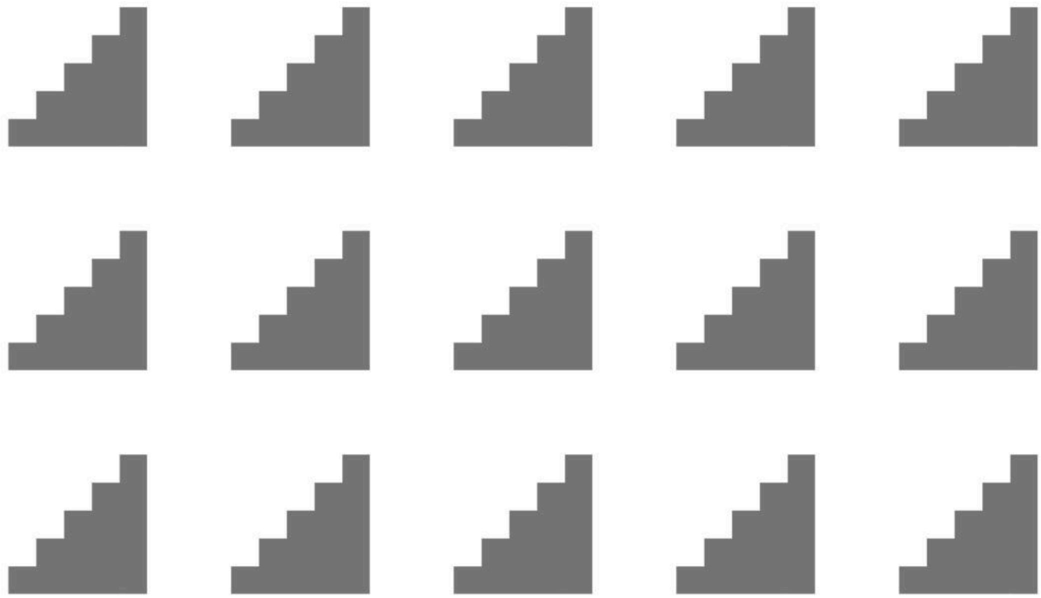
Name: ________________

Stairs	Ways to climb
1	
2	
3	
4	
5	
6	

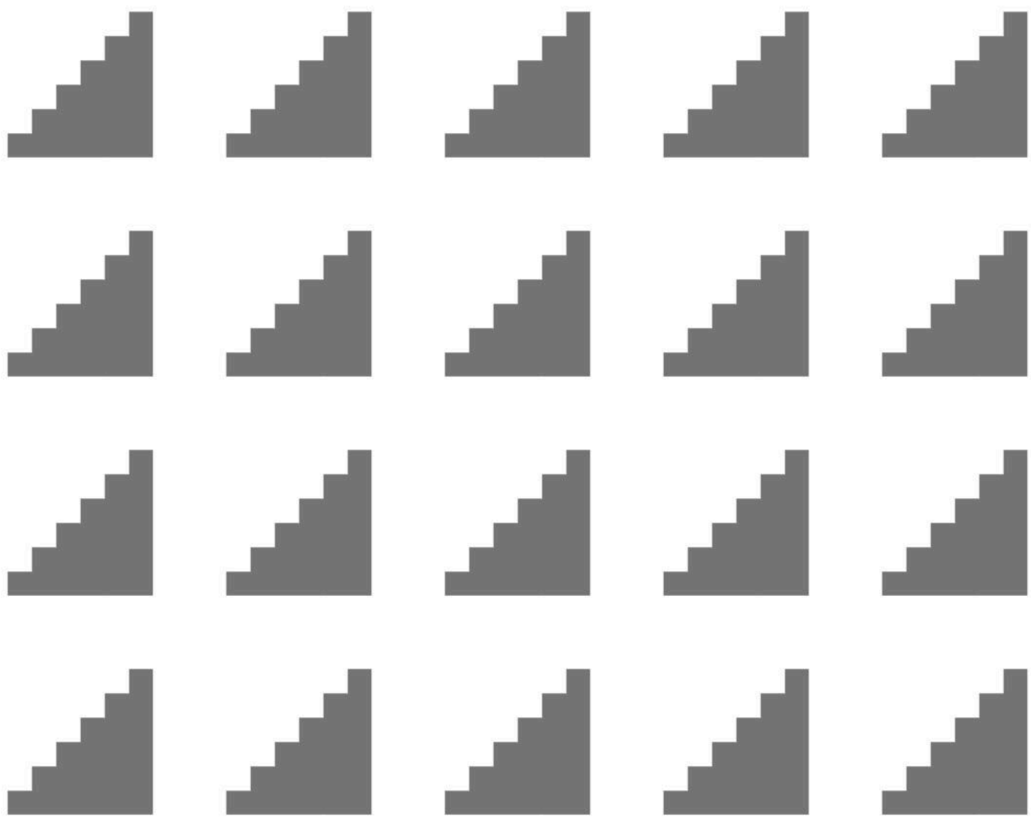
4 stairs:



5 stairs:



6 stairs:

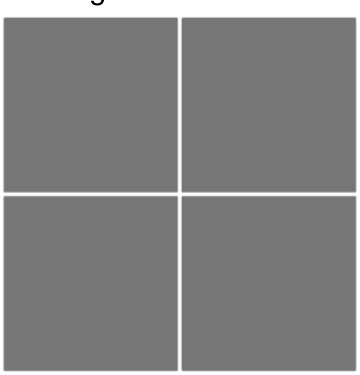


Name: ________________

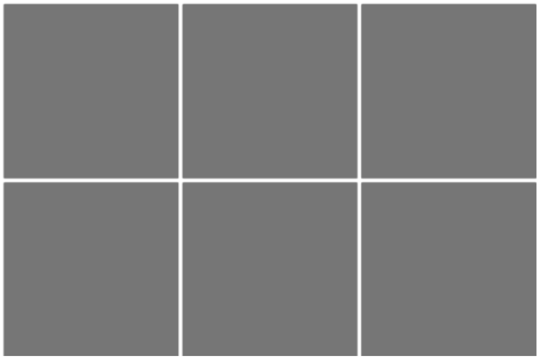
Rectangle	Ways to tile
1	
2	
3	
4	
5	
6	

Rectangle 1:

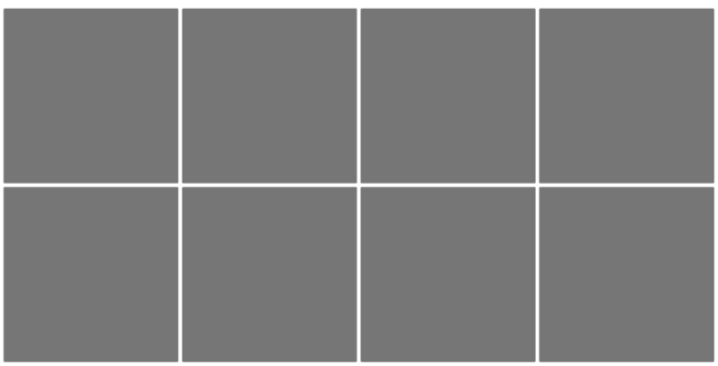
Rectangle 2:



Rectangle 3:



Rectangle 4:



Rectangle 5: