LOGIC

Not And Or

Consider the absurd assertion that

A flea is a mule, or my name is Gerald![1]

That surely must be false![2] *A flea is not a mule and my name is not Gerald!*

- *Which of the following statements are true?*

  - *A flea is a mule and my name is Gerald.*
  - *A flea is a mule and my name is* ____________________.
    fill in your name here
  - *A flea is a mule or my name is* ____________________.
    etc.
  - *A flea is not a mule* **or**[3] *my name is* ____________________.
    your name
  - *A flea is a not a mule and my name is not* ____________________.
    ?? ok mix it up ...

- *Write each of those statements in logical notation.*

- *Complete the truth tables.*
Which expressions are equal to each other? Make up your own!

p	q	$\neg q$	$p \wedge q$	$p \vee q$	$\neg q \vee p$	$\neg p \vee p$	$\neg q \wedge p$	$p \vee (\neg p \wedge q)$	$(\neg p \wedge q) \vee (p \wedge \neg q)$	

We can probably all agree that *"A flea is a mule"* is an absurd, false assertion. And *"My name is Gerald"* is also a false statement[4]. Therefore it must be false that *'A flea is a mule,* or *my name is Gerald!'*

[1]By gum!

[2]...for most of us at any rate. Gerald — fill something else in!

[3]True or false? This depends on our precise meaning of *'or'*. We always mean inclusive *'or'*, meaning one, the other, or both propositions are asserted to be true.

[4]...for most of us!

In logical notation we see ***(Explain why!)*** that in fact:

$$\neg(p \vee q) \quad \text{is logically the same as} \quad (\neg p) \wedge (\neg q)$$

• ***Use a truth table to show that*** $(\neg(p \vee q)) \leftrightarrow ((\neg p) \wedge (\neg q))$ ***is a tautology.***

Consider this assertion:

You're a person and I'm a person.[5]

• ***Write these statements in logical notation.***

We could instead make the false and quite impolite assertion that:

One or both of us is not a person!

But *it is just not true that I am not a person, or you are not a person.*

In fact: $$\neg(p \wedge q) \quad \text{is logically the same as} \quad (\neg p) \vee (\neg q)$$

• ***Use a truth table to show that*** $(\neg(p \wedge q)) \leftrightarrow ((\neg p) \vee (\neg q))$ ***is a tautology.***

Let's see these in action!!

• ***Give another example; give your wackiest:***

p:

q:

And state:

$p \wedge q$ ***vs.*** $\neg p \vee \neg q$
Are these statements each other's negation?

[5]And they're all people too! If only everyone could always remember this...

$\neg p \wedge q$ ***vs.*** $p \vee \neg q$

Are these statements each other's negation?

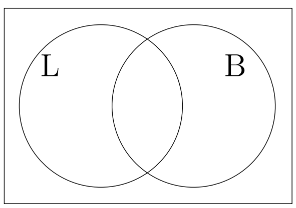
$\neg p \wedge \neg q$ ***vs.*** $p \vee q$

Are these statements each other's negation?

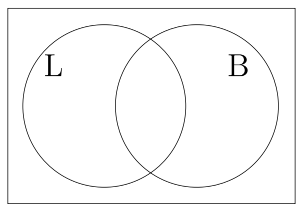
The same thing using Venn Diagrams

Divide the animals up by whether they have four legs (L) and whether or not they are just black and white (B).

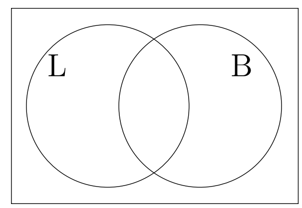
Shade the regions that correspond to the following:



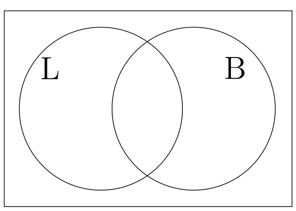
$L \wedge B$



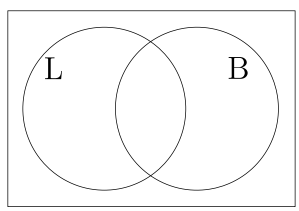
$\neg L \wedge \neg B$



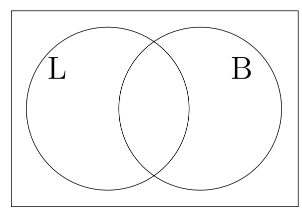
$\neg(L \vee B)$



$L \vee B$



$\neg L \vee \neg B$



$\neg(L \wedge B)$

• ***Add more combinations of*** $\neg$ p q ( ) $\vee$ ***and*** $\wedge$ ***and see what you can produce! Which expressions are equal*** ("logically equivalent") ***to each other? How many truly different kinds of expressions are really possible?***[6] [7]

p	q	

[6]Challenge: show p "not and" q can generate all possible operations on p and q.

[7]Nuther challenge: show a "not and" b can generate all possible operations on $p_1, p_2, \ldots p_k$.

Flying pigs must be purple

Although it is pretty reasonable to presume

If it is snowing then it must be cold.

It really does not seem true that

If it is cold then it must be snowing.

To demonstrate that the statement is false, we have to produce a time when it is cold, but it is not snowing. We can all recall such a time, and we can all agree that the statement is false.

- ***Put the statement and its negation into logical notation.***

It's easier to define what the *negative* of an implication should be: An implication should be false if the hypothesis is true, yet the conclusion is false:

$\neg(p \to q)$ should be defined to be logically equivalent to $p \wedge \neg q$

Consequently

Therefore, we must *define* $p \longrightarrow q$ to be equivalent to $\neg p \vee q$

This is a little surprising! We are *forced* to accept the values you'll fill in the truth table right now; fill in some more!

p	q	$p \wedge (\neg q)$	$\neg(p \to q)$	$(\neg p) \vee q$	$p \to q$		$q \to p$	$p \to \neg q$	$\neg q \to \neg p$	$\neg p \to \neg q$

You will agree that, for every real number x, if $x \geq 1$ then we can be sure that $3x > 2$ (though this bound can be improved!). For example, $7 > 1$ and indeed, $3 \cdot 7 > 2$.

But this statement is not contradicted by values of $x \not\geq 1$. These are and should be irrelevant. For these values of x, the implication is still true regardless of whether or not $3x > 2$.

Consequently, we must accept that

$$(F \to T) = T \text{ and } (F \to F) = T \;!$$

By this rule, absurdities such as the following are perfectly true!!

Flying pigs must be purple.

Logically we are forced to accept that this is a true statement.

In order to demonstrate that this statement is false, we must produce a flying pig that is not purple. There are no flying pigs, much less ones that aren't purple, and there is no counterexample.

The statement cannot be falsified.

Every flying pig really is purple.

In fact, every flying pig is green, red and yellow too, and every flying pig is monochrome, and every flying pig is polychrome! Indeed:

Every flying pig exists! Every flying pig does not exist!

These are both true statements, precisely because no flying pig exists! ***what? why?***

Every logical statement is either true or it is false, and because $p \wedge \neg q$ is false when p is false and q is true, it must be the case that $p \wedge \neg q$ is true when p is true or q is false.

When p is false, $p \to q$ is always true!!	***(Wait what? Why?)***

- ***Fill in these truth tables***

p	q	$p \to q$	$q \to p$	$q \to p$	$\neg q \vee p$	$\neg \wedge p$				

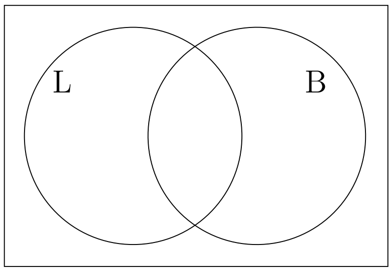
Fill in the remaining entries for all possible combinations of $a \to b$ and $\neg(a \to b)$ with a and b in $p, q, \neg p, \neg q$. Which statements are the same and which are different?

The same thing using Venn diagrams

All of our flying animals (A) have two legs (B).

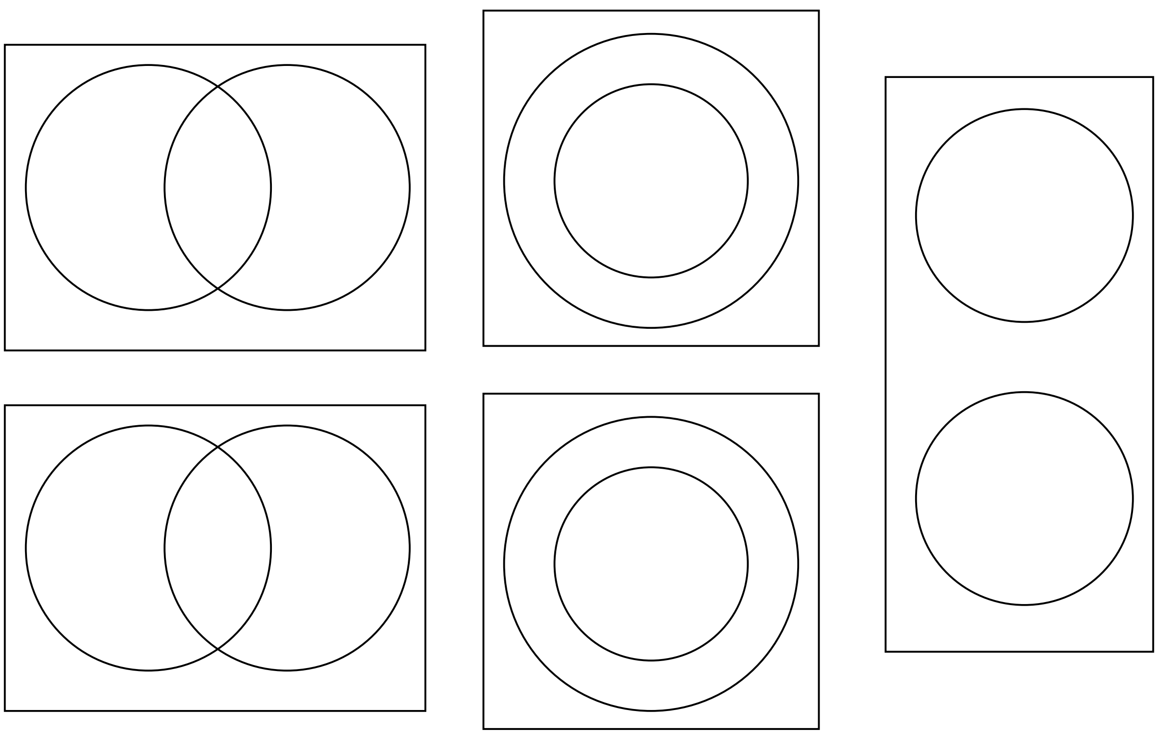
In other words, *If an animal is flying, then it has two legs.*

It is not true that if an animal has two legs it must be flying. What is the *negation* of this statement? One region demonstrates this: *which one?*



Write an expression for this using $\vee, \neg, \wedge$:

Label these in as many ways as possible.
Which regions of these Venn diagrams are occupied, *"If an animal is flying then it has two legs?"*



We really must allow $F \to T$ to be true!

Contrapositively...

We will see that

$$p \to q \quad \text{if and only if} \quad \neg q \to \neg p$$

For example, compare

If someone does not have the money then they cannot buy the crocodile.

vs.

If someone can buy the crocodile then they must have the money.

Clearly these are exactly equivalent assertions. ***(Right? Why?)*** Each is called the <u>*contrapositive*</u> of the other.

- ***Express the statements in logical notation.***

- ***Using a truth table, verify the tautology***

$$((\neg q) \to (\neg p)) \quad \leftrightarrow \quad (p \to q)$$

Because $\neg\neg * = *$, we have a few more contrapositive pairs, such as

$$(\neg p) \to q \text{ and } (\neg q) \to p$$

- ***Write them all out. What do these say about money and crocodiles? Which is most reasonable?***

On the other hand...

Conversely you surely noticed that

$$q \to p \quad \text{and} \quad \neg p \to \neg q$$

are very different than their *converses*

$$p \to q \quad \text{and} \quad \neg q \to \neg p \text{ !!}$$

- *Which of these pairs of statements*[8] *makes more sense?*

"If you do not have the proper security clearance, then you may not enter the facility."

&

"If you may enter the facility, then you must have the proper security clearance."

vs.

"If you have the proper security clearance, then you may enter the facility."

or

"If you may not enter the facility, then you do not have the proper security clearance."

For each of these statements, what is its

- *contrapositive*
- *converse*
- *negation*
- *negation of the converse contrapositive*

1. *If a baby then can manage a croc.*
2. *If not a baby then can't manage a croc.*
3. *If can manage a croc then not a baby.*
4. *If can't manage a croc then a baby.*

[8](Obviously read out in a metallic monotone.)

If ‘if’s and ‘but’s were candy or nuts
we’d all have a very merry Christmas!

Ok think about it. For the title, and for each of the following,

1. give the
   (a) contrapositive,
   (b) converse,
   (c) negative,
   (d) and converse contrapositive,
2. check in logical notation, and
3. make a Venn diagram!

- *If a goose says hello, then it sure is a Tuesday!*
- *If the sky is green, this must be Planet Grek62 or Planet &w^{6}Q7q.*
- *The adventurers will prevail only if she has brought a paperclip and he has the thumbtack.*
- *If* $x > x^2$ *then* $0 < x < 1$. *(Is this true for all* $x \in \mathbb{R}$*?)*[9]
- *If* $x^2 > 0$ *then* $0 < x$ *or* $x < 0$. *(Is this true for all* $x \in \mathbb{R}$*?)*
- *If* $x \in \mathbb{R}$ *then* $x + 1 > x$. *(ditto)*
- *If* $x \in \emptyset$ *then* $x > x + 1$. *(ditto)*
- *If* $x^2 + 2x = 0$ *then* $x = 0$ *or* $x = -2$. *(...)*
- *If* $x \neq 2$ *and* $x \neq 0$ *and* $x \neq -2$ *then* $x^2 \neq 2x$.

Make up a few more of your own!

[9] Hint: “$0 < x < 1$” $\leftrightarrow$ “$0 < x$” and “$x < 1$”.

Therefore

Some nice warm-ups!

- `Give truth tables for`

1. $p \wedge q \wedge r$
2. $p \vee q \vee r$
3. $(\neg p) \vee (\neg q) \vee (\neg r)$
4. $(p \vee (q \wedge r)) \wedge (q \vee r)$
5. $(p \wedge q) \vee (q \wedge r) \vee (r \wedge p)$
6. .

- `Verify`[10] `each of the following tautologies.`

1. $\neg\neg p \leftrightarrow p$
2. $p \wedge \neg p \leftrightarrow F$, $p \vee \neg p \leftrightarrow T$
3. $p \wedge T \leftrightarrow p$, $p \vee F \leftrightarrow p$
4. $(p \vee q) \rightarrow (p \wedge q)$
5. $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$
6. $\neg(p \rightarrow q) \leftrightarrow (p \wedge \neg q)$
7. $(\neg(p \vee q)) \leftrightarrow ((\neg p) \wedge (\neg q))$
8. $(\neg(p \wedge q)) \leftrightarrow ((\neg p) \vee (\neg q))$
9. $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
10. $p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$
11. $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$
12. $(p \wedge (p \rightarrow q)) \rightarrow q$
13. $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$
14. $((\neg q) \wedge (p \rightarrow q)) \rightarrow \neg p$
15. $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \wedge r)$
16. $(\neg(p \rightarrow (\neg q))) \leftrightarrow (p \wedge q)$
17. $((\neg p \wedge q) \rightarrow \neg p)$
18. $(((p \wedge q) \rightarrow \neg p) \leftrightarrow \neg(p \wedge q))$
19. $\neg p \rightarrow ((p \wedge q) \rightarrow \neg q)$
20. $((p \wedge q) \rightarrow \neg q) \rightarrow q$

- `Check them using truth tables!`
- `Pick wacky` p `and` q `and see if these make sense!`
- `Make up new and clever examples!`

- `Express these as set identities and check them using Venn diagrams`

- `Express these as syllogisms.`

[10] (or fix)

Some helpful truth-table paper.

What operations are even possible?

`How many operations are possible?` There are 2^2 lines in the truth table for two propositions, say p and q. How many *different* operations are possible, including the boring ones T, F, p, q?

What about for three propositions? Four?

`Challenge puzzle!` Some operations can be written in terms of others: For example, if you already have $\vee$ and $\neg$, you don't really need the operation $\rightarrow$, because

$$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$$

Or, for that matter if you already had $\rightarrow$ and $\neg$, then you wouldn't really need $\vee$, since

$$(p \vee q) \leftrightarrow (\neg p \rightarrow q)$$

What are the smallest sets of operations that will give all the others? Is it possible to make all the others out of *just one* special operation?

`Double Challenge!` With three propositions, there are 2^{2^3} different operations. Can each of these be formed by stringing together operations on two propositions at a time? Or are there truly "three-dimensional" operations? What about "four-dimensional" operations, etc.?

It depends

<u>*Predicates*</u> are phrases like these, that aren't really *True* or *False* until we know what they refer to: the specific value of x, the name of a person, or what planet we're on.[11]

- $x > 7$ (the sentence *"x is greater than 7"*)
- *Their favorite ice cream is strawberry.*
- *A flea is a mule or her name is Gerald!*
- *It happened on Wednesday.*
- *That planet is Planet Grek62 or Planet &w^{6}Q7q*
- *Make up more!*

A predicate $p(x)$ might or might not be true, depending on the value of x.

We can *quantify* a predicate $p(x)$ by stating

For all values of x, $p(x)$ is true.

For some value of x, $p(x)$ is true.

[11]A predicate is really a *predicate function* $p : U \to \{T, F\}$, for each possible value x_0 in some universal set U, p returns the truth or falsehood of the proposition $p(x_0)$.

Somebody just gotta love somebody

1. Let $L(x, y)$ be the predicate *"x loves y"*. Write each of the following in symbolic form:

   (a) Someone loves everybody.

   (b) Someone loves someone.

   (c) Everyone loves somebody.

   (d) Everyone loves everybody.

   Write the negation of each of the above symbolically, and in words.

2. Which of the following are true? Rewrite each statement in symbolic form. Write the negative of each statement.

   (a) There is an integer n such that $n^2 \leq 0$.

   (b) For all integers n, $n^2 \geq 0$.

   (c) For any n, there is some integer that is greater than n.

   (d) There is an integer n such that all integers are greater than or equal to n.

3. For each of the following, decide whether the statement is true or false, and give a short explanation. (Keep in mind: What is the standard of proof in each case?)

   In every case, the domain of discourse is $\mathbb{R}$, the set of real numbers.

   $\exists x \in \mathbb{R},\ x^2 = 2$

   $\forall x \in \mathbb{R},\ x^2 = 2$

   $\forall x \in \mathbb{R},\ x^2 > 4 \to x > 2$

   $\exists x \in \mathbb{R},\ x^2 < 0 \to 1 = 2$

   $\forall x \in \mathbb{R},\ x > 1 \to x^2 > x$

   $\exists x \in \mathbb{R},\ x > 1 \to x^2 > x$

   $\forall x \in \mathbb{R},\ x > 0 \to x^2 > x$

   $\exists x \in \mathbb{R},\ x > 0 \to x^2 > x$

4. For each of the following, decide whether the statement is true or false, and give a short explanation. (Keep in mind: What is the standard of proof in each case?)

   In every case, the domain of discourse is $\mathbb{R}$, the set of real numbers.

   $\forall x \in \mathbb{R},\ \forall y \in \mathbb{R},\ x = y + 3$

   $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ x = y + 3$

   $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ x = y + 3$

   $\exists y \in \mathbb{R},\ \exists x \in \mathbb{R},\ x = y + 3$

   $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ y > x$

   $\exists y \in \mathbb{R},\ \forall x \in \mathbb{R},\ y > x$

   $\exists x \in \mathbb{R}, \forall y \in \mathbb{R},\ xy = x$

   $\exists x \in \mathbb{R}, \forall y \in \mathbb{R},\ xy = y.$

   $\forall x \in \mathbb{R}, (x \neq 0) \to (\exists y \in \mathbb{R}, xy = 1)$

1. It is false that

   Nobody who is not a baby can't manage a crocodile.

   Its negation is true:

   Some people who are not babies can't manage a crocodile.

   Write these in logical notation.

2. What is the relationship of that statement and

   Nobody who can't manage a crocodile isn't a baby.

   and

   Nobody who can manage a crocodile is a baby.

   Draw these as Venn diagrams.

3. What is the negation of the statement $\forall a, \exists b,$ s.t. $a \neq 0 \rightarrow ab = 1$

4. Over the domain of discourse $\mathbb{R}$, determine if the following statements are true or false, and justify your answers.

   (a) $\forall x, x > x - 1$

   (b) $\forall x \forall y, (x + y = 4) \rightarrow (x = y)$

   (c) $\forall x \exists y, (x + y = 0) \rightarrow (x - y = 0)$

5. Give a formal proof that the following is true:

   $$\forall x \in \mathbb{R}, (x \neq 0) \rightarrow (\exists y \in \mathbb{R} \text{ s.t. } xy = 1)$$

6. For the statement

   $$\forall x \exists y \text{ s.t. } xy = 1$$

   (a) Give the negation of the statement.

   (b) Formally prove that the negation is true (hint: 0) and the statement is therefore false.

7. Are the following true or false? Justify your answer.

   The domain of discourse is the real numbers $\mathbb{R}$.

   (a) $\forall x, x^2 > x$

   (b) $\forall x, x > 1 \rightarrow x^2 > x$

   (c) $\forall x, \exists y$ s.t. $x > y$

   (d) $\exists y$ s.t. $\forall x, x > y$