Dear attendants of MoMath' 'Ask a Mathematician Anything!' on Thursday June 6:

No sooner had our meeting ended, that I felt bad about not having answered satisfactorily, and in (fairly) elementary terms, the question for an explanation that the complex numbers are algebraically closed. In other words, showing that every polynomial of degree n, with complex coefficients, has n complex roots (counting multiplicity).

The only (feeble!) excuse I can think of is that I was tired and would have needed 10 minutes of staring into the air (something to which a Zoom meeting of 1 hour doesn't lend itself well) to reconstruct the argument, and I didn't have it at my fingertips because it is over 1/3 of a century ago since I last taught complex analysis.

But of course, within 10 minutes after the Zoom ended, I had done that staring into the void, and reconstructed the argument (without looking it up -- promise!) , without the fancy hocus-pocus that I had invoked during the meeting, as indeed any mathematician worth their salt can. Here is that argument.

So suppose we have a complex polynomial of degree n, in the variable z. Because the coefficient of the n-th power of z is non-zero (otherwise the polynomail would be of a lower degree!), we can divide the whole polynomial by this complex coefficient, resulting in the polynomial p(z) of the form

$$p(z) = z^n + \sum_{j=1}^{n} c_j z^{n-j}$$

where the remaining other coefficients are complex numbers, some of which could be (but need not be) zero. If they are all zero, then p(z) reduces to the n-th power of z, which has a zero (with multiplicity n) at z=0, and we have nothing to prove. So let's assume that at least one of the coefficients is non-zero.

The first observation we are going to make is that we can derive an easy bound on the modulus of any possible zero of p(z), as follows:

if $p(u)=0$, then $$u^n = -\sum_{j=1}^{n} c_j u^{n-j}$$

$$\Rightarrow \quad |u|^n = \left|\sum_{j=1}^{n} c_j u^{n-j}\right| \leq \sum_{j=1}^{n} |c_j|\,|u|^{n-j}$$

if $|u| \leq 1$, then we have a bound on $|u|$ already, and we are done.

if $|u| > 1$, then $|u|^{n-j} \leq |u|^{n-1}$ for each $j \in \{1, \dots, n\}$, so that

$$|u|^n \leq |u|^{n-1} \sum_{j=1}^{n} |c_j|,$$

which implies $$|u| \leq \sum_{j=1}^{n} |c_j| .$$

So any u for which $p(u)=0$ must satisfy $$|u| \leq \underbrace{\max\left(1, \sum_{j=1}^{n} |c_j|\right)}_{\text{denote this by } R.}$$

We can use this bound R for other purposes too:

if $|z| > 2R$, then

$$|p(z)| \geq |z|^n - \sum_{j=1}^{n} |c_j|\,|z|^{n-j}$$

$$= |z|^n \left(1 - \sum_{j=1}^{n} |c_j|\,|z|^{-j}\right)$$

$$\geq |z|^n \left(1 - \sum_{j=1}^{n} |c_j|\,|z|^{-1}\right)$$

$$\geq |z|^n \left(1 - R/|z|\right) \geq |z|^n \left(1 - \frac{1}{2}\right) = \frac{1}{2}|z|^n$$

This statement is just a way of expressing that outside some (possibly huge) circle, the largest power-term in z dominates the behavior of the polynomial p(z).

It also implies that for all z outside that same huge circle, the function g(z)= 1/p(z) is well defined and bounded:

$$|g(z)| \leq \frac{1}{|p(z)|} \leq 2|z|^{-n} \leq 2^{1-n} R^{-n} \leq 1 \quad \text{for } |z|>2R \quad (\text{we have used that } R \geq 1)$$

(Note that the inequality in the middle also implies that g(z) tends to zero as the modulus of z grows to infinity

Now if p(z) had also no zeroes inside that circle, then g(z)=1/p(z) would be a perfectly nice continuous complex function defined everywhere.

At this point we have to invoke a few facts about continuous functions:

1. A complex function h(z) that is well-defined and continuous in z on a disk in the complex plane (i.e. for all z with modulus smaller than some T) is always bounded -- that is, there exists some (possibly huge) number K (which depends on h and on T) such that the modulus of h(z) cannot exceed K for any z in that disk.

2. A complex function h(z) that is defined on the whole complex plane and continuous at every point in the complex plane can be uniformly bounded on the complex plane only if it is constant.

[The first fact is something that is true also for real functions of real variables, with 'disk' replaced by 'interval'; in fact it is a very general statement about the behavior of continuous functions on what are called 'compact sets' in analysis.
The second fact is a very special fact about complex continuous functions, and can be proved as a consequence of the Cauchy integral formula, a basic building block in complex analysis.]

Together with our earlier observations, these two facts imply that p(z) MUST have a zero:

* g =1/p has no zero outside some big disk, and is bounded outside that disk;
* if p has no zero inside the disk, then g=1/p is well-defined inside the disk, and continuous, and thus bounded inside the disk (by fact 1);
* that means that g would then be a continous and bounded function on the whole complex plane, so that it must be a constant (by fact 2):
* but we also know (see the green obeservation above) that g(z) tends to zero at infinity
* this means that g(z)=0 for all z (since g must be constant), which is an absurdity.

Since the assumption that $p(z)$ has no zero at all leads to a contradiction, p must have at least one. So let us assume that $p(a)=0$.

Given any polynomial $r(z)$ of degree n, and any complex number b, we can always find (by solving a linear sytem to find its coefficients) a new polynomial $s(z)$, of degree $(n-1)$ so that

$$r(z) = (z-b)\, s(z) + c$$

where the remainder c is in fact equal to $r(b)$.

If we now do that for the polynomial $p(z)$, and choosing a for the complex number b, then we obtain

$$p(z) = (z-a)\, q(z)$$

(the remainder equals $p(a)=0$ now), and we can repeat the argument for the polynomial q of degree $(n-1)$ to 'peel off' further zeroes of p.

(Slightly more sophisticated, one can make this an argument by induction on n, of course. If you know what an argument by induction is, you can easily see how. If not, then the 'peeling off', going down by one degree until you are at the end, works too -- and you do get n zeroes, some of which may be the same one, repeated a number of times.)