# MO彞MATH 

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## Cutting out Hearts Kathy Paur



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## 1 Introduction and Handout

The handout that follows is a sequence of puzzles in which students fold paper in increasingly complex ways so that they can cut exactly one arc to create a pattern of heart-shaped holes in their paper.

Many of us learn in early elementary school how to fold a piece of paper in half, cut along an arc, and produce a Valentine's Day heart. You can probably imagine how to use some extra folds to cut out a whole grid of hearts - but what if we break the symmetry by leaving one heart out of the grid? Can you make silly "mistakes" on purpose?
"Cutting out Hearts" works well for students from kindergarten through eighth grade, as well as puzzle-loving adults. Because this activity is a puzzle sequence, it differentiates easily; different grades and students start with different puzzles, and work faster or slower without compromising the lesson.

Symmetry is the key mathematical concept illustrated in "Cutting out Hearts." Along the way, students learn to exploit symmetry in designing their solutions, and they see how much complexity is required to solve puzzles with broken symmetry. Additionally, they realize how a simple activity like cutting out a Valentine's heart can yield an increasingly complex and devious puzzle sequence, and they practice reverse engineering, flexible thinking, and persistence.

The handout has 10 pages, not all of which are meant to be used simultaneously:

- Page $i$, Instructions
- Page ii, Puzzles A-F
- Page iii, Puzzles G-L
- Page iv, Puzzles M-R
- Page $v$, Puzzles S-X
- Page vi, Puzzles Y-Z
- Page vii, Make your own puzzles (big kids)
- Page viii, Make your own puzzles (big kids)
- Page ix, Make your own puzzles (little kids)
- Page $x$, Blank puzzles page (for teachers to add puzzles)


## Cutting Out Hearts

Starting with old scratch paper or standard printer paper, create each of the following designs by making as many folds as you like and cutting exactly one half-heart-shaped arc, like this:


Check off the puzzles as you solve them. They get increasingly fun and difficult!
$\square \mathrm{A}$

C

$\square$ F



$\square \mathrm{N}$

$\square \mathrm{P}$

$\square \mathrm{R}$


$\square$ T

$\square$ U
$\square \mathrm{V}$

$\square \mathrm{W}$

$\square \mathrm{X}$

$\square \mathrm{Y}$

$\square \mathrm{Z}$


## Try making your own puzzles!

Make some that you think would be fun but not super difficult:



Now make some that you think are hard but not impossible:
$\square$
$\square$

Finally, try to make some that are impossible! (Don't make them supercomplicated though. It's more fun if they only have a few hearts, but still turn out to be impossible.)



Now try a friend's puzzles!

## Make your own puzzles!





$\square$

Now try a friend's puzzles!
$\square$ $\square$

$\square$


## 2 Implementation

### 2.1 Before Class

### 2.1.1 Understanding the Activity

Solve some of the puzzles yourself! It's okay if you don't do very many of them. Here's a quick breakdown of puzzle difficulty:

- Many adults can solve puzzles A through F in their heads.
- Puzzles G through L are fairly straightforward, but it can help to actually do them with paper and scissors to understand what you're working with.
- Puzzles M though P get more difficult, but are generally accessible.
- Puzzles Q through T require significant work to understand how to "move paper out of the way."
- Puzzles U through W are quite difficult.
- As of September 2023, puzzle X is unsolved! ${ }^{1}$
- Puzzles Y and Z are unsolved, and may be of use in investigating the conjecture in Section 5.3 or finding a stronger result. They aren't recommended for classroom use.

The teacher not knowing how to solve all the puzzles is a fabulous motivator for students, and is strongly recommended.

### 2.1.2 Budgeting Time: How long should the activity take?

Of course this depends on your schedule and student routines, but here are some suggestions:

- Kindergarten: 30-45 minutes
- Grades 1-2: 45 minutes
- Grades 3-5: 45-60 minutes
- Grades 6-8: 60-75 minutes

If you want to implement any of the extension activities in Section 3, you may want to plan for additional time.

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### 2.1.3 Materials

Prep your supplies:

- Paper: about 25 sheets of scrap paper or printer paper for each student. This can be old scrap paper with writing on both sides!
- Scissors: one pair per student.
- Pencils and Markers: one each per student. (Just have these available. They may be useful for some students midway through the activity.)
- Puzzle Copies: one set per student. (Grade-level differentiation suggestions below).
- Recycling Bin: students go through a lot of paper!


### 2.1.4 Differentiation by Grade

These puzzles can stretch all the way from kindergarten through 8th grade. The last few puzzles can be challenging even for adults! Here are some suggestions about the range of puzzles to use by grade, though of course some students will be better off starting with easier puzzles or skipping some puzzles.

| Grade | Copy | Suggested Approach |
| :---: | :---: | :---: |
| K | - Puzzle Page $i i$ (puzzles A-F) <br> - Puzzle Page iii (puzzles G-L) <br> - Puzzle Page ix (make your own puzzles, no additional directions) | - Give everyone puzzles A-F to start <br> - Give them puzzles G-L when/if they've completed puzzles A-F (or if they just want them). (Many students will not need puzzles G-L.) <br> - Give the make-your-own puzzles page to students who seem fidgety, or to everyone if you prefer - it just depends what you want to emphasize. |
| 1, 2 | - Puzzle Page $i i$ (puzzles A-F) <br> - Puzzle Page iii (puzzles G-L) <br> - Puzzle Page $i v$ (puzzles M-R) <br> - Puzzle Page $i x$ (make your own puzzles, no additional directions) | - Give everyone puzzles A-L to start <br> - Keep puzzles M-R in reserve in case anyone finishes A-L, or just wants something different to work on. (Most students will not get past puzzle L.) <br> - Give the make-your-own puzzles page to students who seem fidgety, or to everyone if you prefer - it just depends what you want to emphasize. |


| Grade | Copy | Suggested Approach |
| :---: | :---: | :---: |
| 3, 4 | - Puzzle Page $i i$ (puzzles A-F) <br> - Puzzle Page iii (puzzles G-L) <br> - Puzzle Page iv (puzzles M-R) <br> - Puzzle Page $v$ (puzzles S-X) <br> - Puzzle Pages vii and viii (make your own puzzles, fun/tough/impossible) | - Give everyone pages $i i$ and $i i i$ to start, and suggest that they do puzzles $\mathrm{B}, \mathrm{D}$, E , and F as a warmup, and continue on from G. <br> - Give pages $i v$ and $v$ to anyone who finishes puzzles A-L (or mostly finishes them, and wants to see the remaining puzzles). Explain that these pages are much trickier. <br> - Use the make-your-own puzzles pages at your discretion - either give them individually to any students who seem like they would benefit, or turn them into a class puzzle-making activity. |
| $\begin{aligned} & 5,6, \\ & 7,8 \end{aligned}$ | - Puzzle Page $i$ (instructions) <br> - Puzzle Page $i i$ (puzzles A-F) <br> - Puzzle Page iii (puzzles G-L) <br> - Puzzle Page $i v$ (puzzles M-R) <br> - Puzzle Page $v$ (puzzles S-X) <br> - Puzzle Pages vii and viii (make your own puzzles, fun/tough/impossible) | - Hand out packets. <br> - Explain that the first two pages are fairly straightforward, but they should do puzzles G, H, and J as a warmup. <br> - If they want, they can do earlier puzzles as well, but this isn't required. <br> - When a student feels warmed up, they should move on to puzzle K and go from there. <br> - Students will find the make-your-ownpuzzle pages on their own, and may choose to engage with them. Most students tend to stick with the puzzles for at least an hour. <br> - If you wish, run a make-your-ownpuzzle extension activity. |

### 2.2 During Class

### 2.2.1 Introducing the Puzzles

For kindergarteners through second graders, teach them to make hearts from folded paper if they don't already know how! Then explain that it's their turn to make hearts in more complicated designs. Hand out the puzzle sheets and supplies, and explain that they should generally solve the puzzles in order - though if they get stuck, it's okay to ask for help or try a different puzzle.

Emphasize that students can make as many folds as they like, but they may only cut one half-heart! Younger students tend to miss this if it's not reiterated.

For older kids, start by reminding them how they probably learned to cut hearts from folded paper in early elementary school. Explain that this simple activity can become deviously challenging for more complex heart designs. Then hand out supplies and suggested puzzles (see grade-level differentiation suggestions below). Again, reiterate, "you can make as many folds as you want, but only cut one half-heart!"

Explain that the puzzles become progressively harder. If there are puzzles you don't know how to solve, tell the students, and express confidence that some of them may figure those puzzles out and show you!

Almost silent introduction: Alternatively, you can introduce the activity just with demonstration. Hold up a sheet of paper, fold it in half, and cut out a single heart. As you cut, students will realize what you're making and comment on it. Then hold up a second sheet of paper, fold it, and cut from the wrong side to make a broken heart (like Puzzle B). Students will realize the mistake as you cut and whisper, smile, or comment. Hold up a third paper sheet, make two or three random folds, and cut a half-heart arc along a fold edge. Unfold it to show the students your silly design. Then hand out puzzles and supplies, and proceed as above.

Optional experimentation phase: Students may find it fun and helpful to engage in free play before moving on to the puzzles, by folding their paper any way they like, and cutting one half-heart. This approach gives students a sense of many possible approaches quickly, and reinforces the value of open-ended play in mathematical thought. It does require the teacher to refocus the class on puzzles after free play, so skip it if you want to avoid the transition time.

### 2.2.2 During the Activity.

Sometimes students get stuck! In many of these cases, it can help to grab a pencil and ask them to draw the lines of symmetry for each heart on the puzzle paper; sometimes it also helps to ask them to draw the additional fold lines they think they'll need.

Occasionally a younger student doesn't want to stick with solving prescribed puzzles, and this can be a good time to use the "make your own puzzle" pages.

Once older students begin working on the trickier puzzles, they often draw hearts on their paper so that they can keep track of heart positions as they try complex folds. (This works especially well when they use marker that bleeds through the paper, so that they can see the hearts from both sides.) This is a fabulous approach, but it may be better not to suggest it initially. The spatial skills required to start with blank paper are worth honing, and this can be a good opportunity to let students realize that they've independently found a valuable way to improve their problem solving.

Collaboration is at the teacher's discretion. The author encourages it through table seating, and by giving enthusiastic consent if students ask about working together. However each teacher knows their students and what works for them in their environment.

### 2.3 Discussion and Takeaways

The most important part of the lesson is in the cutting and thinking and peer discussion. However, it is worthwhile to make sure that a student articulates for the class the basic point that fold lines create reflections.

Teachers may wish to stop the students for a few short discussions, which could include:

- What strategies have you used? (This is a good time to have students hold up puzzle sheets where they've drawn fold lines, or scrap paper where they've drawn the hearts.)
- Can you tell just by looking at a puzzle how many fold lines you will need?
- What makes a puzzle hard? (Or, in case of blank looks, "What makes a puzzle easy?")
- (Older kids only) Do you think there are any impossible puzzles? What would make a puzzle impossible?
- Closer: Remember, we started with a simple kindergarten activity, and eventually found questions we couldn't answer.

In addition, the lesson can lead into a number of extension exercises which are detailed in the following section.

Finally we would be remiss not to mention that this is an excellent time to watch Vi Hart's video on Snowflakes, Starflakes, and Swirlflakes. Older students should be challenged to understand Vi Hart's internet argument one minute into her Hexaflexaflakes video.

## 3 Extension Exercises

### 3.1 Extension: Make Your Own Puzzles

"Cutting out Hearts" naturally extends into a make-your-own-puzzle exercise, which can be used in two ways:

- at the individual level, to refocus students or give them a different kind of challenge.
- class-wide, to help students think through the characteristics of easier or harder puzzles. This can lead to a discussion of which puzzles are in fact impossible.


### 3.1.1 Younger Students

For younger students (K-2), the simpler make-your-own-puzzle sheet on page $i x$ suffices. Encourage them to draw their own puzzles of any kind.

Ask each student to pick one of their own puzzles that they think they can solve, and draw it in dark marker on a full sheet of paper. Post all these puzzles on a wall or board, and ask students to solve them.

### 3.1.2 Older Students

Older students can think through the aspects of a good puzzle. They recognize that symmetry makes a puzzle easier, and too much broken symmetry (or too many hearts) can make a puzzle no fun. They can benefit from the guidance in the make-your-own-puzzle sheets on pages vii and viii. These pages suggest students think about the three categories of puzzles in Figure 1.

Once students have made their own puzzles, and possibly tried out their friends' puzzles, have them choose one puzzle to share with the class. Post these on a board or wall, divided into sections labeled "Fun," "Tough But Probably Possible," and "Probably Impossible." (See Figure 2.)

This is a good time for a discussion about what the students think makes a puzzle impossible, as well as the observation that it's remarkable that they started with a kindergarten craft exercise and ended up considering questions of possibility or impossibility. For more discussion on impossible puzzles, see Section 5 .

If preferred, students' puzzles can be transferred to copies of handout page $x$ for a classpersonalized puzzle list.

| Puzzle Type | This should be a puzzle that: |
| :--- | :--- |
| Make Your Own <br> Fairly Easy Puzzle | You think you could do without too much trouble. Ideally, your class- <br> mates will agree that it's fun and not too hard. |
| Make Your Own <br> Difficult Puzzle | You are pretty sure is solvable, even if it's difficult. You should prob- <br> ably only use 2 or 3 hearts, and they shouldn't be too close together. <br> Too much symmetry will make your puzzle too easy, though some <br> symmetry can be okay. |
| Make Your Own <br> Impossible Puzzle | You're pretty sure is impossible, maybe because the hearts are too <br> close together. It's easy to make apparently-impossible puzzles that <br> have 20 or more randomly scattered hearts, but much more elegant <br> and compelling to make apparently-impossible puzzles with only 2 or <br> 3 hearts. In fact, it's worth trying to think of the simplest puzzle you <br> can that you're pretty sure is impossible. |

Figure 1: Puzzle Types!


Figure 2: Make your own puzzles!

### 3.2 Extensions: Connecting "Cutting out Hearts" to Content Standards

"Cutting out Hearts" can be used to support or introduce content standards from kindergarten through high school. This happens naturally in grades 4 and 8, and in high school geometry, where symmetry is an explicit part of the common core standards. However, there are a number of activities that connect "Cutting out Hearts" to other standards involving fractions and algebraic expressions.

### 3.2.1 Kindergarten: Language

"Cutting out Hearts" is a great opportunity to incorporate language about position and dimension, as suggested in the following content standards:

CCSS.MATH.CONTENT.K.G.A.1. Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as above, below, beside, in front of, behind, and next to.

CCSS.MATH.CONTENT.K.G.A.3. Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").

### 3.2.2 Grades 1-3: Fraction Connections

The rectilinear folds required for Puzzles A through I have significant similarity to the pictures we create to teach fractions in the lower grades, as in Figure 3. We can use this to reinforce ideas of fraction and correspondence.
3. Color $\frac{1}{2}$ of each shape.
(a)

(b)
(c)
(d)

4. Color $\frac{1}{4}$ of each shape.
(a)

(b)
(c)
(d)

2. Color a fourth of each of the following shapes.

(1) Making Halves and Fourths

Fold a piece of square paper into halves.
Then fold it into fourths.


1. Write the correct fraction for each shape.


Figure 3: Early elementary fractions exercises, taken from Singapore Math Standards Edition, 1 B and 2 B .

To connect "Cutting out Hearts" to these early elementary fraction exercises, draw the folds needed for Puzzles A through I, as in Figure 4, and note how they divide the paper into halves, thirds, or quarters, with each fractional part being a congruent rectangle, and each fractional part containing one half-heart. Then look through the rest of the puzzles to find other designs that can be divided into congruent parts that each contain a heart or half-heart. For example, Puzzle T can be divided into two triangles each containing a heart, or four rectangles each containing a half-heart. Draw students' attention to the similarity with visual fraction exercises they've seen elsewhere.
(The fractions associated with Puzzles A-F will be appropriate for 1st grade. Second grade can add Puzzle G, and third grade can work with puzzles A-I. Once you've seen how your students respond, you can see which other puzzles you might want use.)


Figure 4: Building fraction representations with rectilinear fold lines. This can also be done with circular paper and rosette symmetry.

This activity aligns with the following common core standards:
CCSS.MATH.CONTENT.1.G.A.3. Partition circles and rectangles into two and four equal shares, describe the shares using the words halves, fourths, and quarters, and use the phrases half of, fourth of, and quarter of. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

CCSS.MATH.CONTENT.2.G.A.2. Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

CCSS.MATH.CONTENT.2.G.A.3. Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words halves, thirds, half of, a third of, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

CCSS.MATH.CONTENT.3.G.A.2. Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. For example, partition a shape into 4 parts with equal area, and describe the area of each part as $1 / 4$ of the area of the shape.

### 3.2.3 Grade 4: Practicing with Geometry Language and Tools

The 4th grade common core standards lay most of the foundation for Euclidean geometry and measurement: angles and their measures, parallel and perpendicular lines, symmetry. The unfolded solutions left over from "Cutting out Hearts" provide straightforward practice with these concepts. Teachers can ask their students:

- Which fold lines are perpendicular?
- Which are parallel?
- What angles do you see in the fold lines for Puzzle C? J? M?
- To do a perfect job folding Puzzle M, use a protractor and draw your fold lines first.
- What if you wanted to make a rosette of hearts with 5 -fold symmetry? Can you do the division, and use a protractor to draw the fold lines on paper?

This type of work aligns with the following common core standards:
CCSS.MATH.CONTENT.4.MD.C.5. Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:
a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through $1 / 360$ of a circle is called a "one-degree angle," and can be used to measure angles.
b. An angle that turns through $n$ one-degree angles is said to have an angle measure of $n$ degrees.

CCSS.MATH.CONTENT.4.MD.C.6. Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.

CCSS.MATH.CONTENT.4.G.A.1. Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

CCSS.MATH.CONTENT.4.G.A.2. Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

CCSS.MATH.CONTENT.4.G.A.3. Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry

### 3.2.4 Grades 4-7: Finding Patterns, Writing and Evaluating Expressions

In grades 4-7, the common core standards emphasize analyzing patterns and relationships, and using expressions to represent these. For example, the dot patterns in Figure 5 encourage students to find a pattern, represent it with an algebraic expression, and use their expression to predict the number of dots in a much larger configuration.

## PRACTICE Answer the following questions about the gumball pattern below.



Figure $1 \quad$ Figure 2

figure 3
128. How many gumballs are needed to make the $8^{\text {th }}$ figure in this pattern?
129. Write a simplified expression for the number of gumballs needed to make the $n^{\text {th }}$ figure in this pattern.
130. There are 221 gumballs in the $k^{\mathrm{th}}$ figure of this pattern. What is $k$ ?
128.

129.

130. $k=55$

Figure 5: A standard find-the-pattern-write-an-expression problem from Beast Academy 5C.
"Cutting out Hearts" gives rise to similar pattern and expression questions, as shown in Figures 6 and 7.


Figure 6: How many fold lines are needed to cut out $h$ hearts, if we restrict ourselves to vertical folds?


Figure 7: If we repeatedly fold the paper in half, alternating vertical and horizontal folds, how many hearts can we cut out? How many fold lines will we create? (This second question is substantially more challenging.)

Students and teachers can create similar pattern-and-expression questions to investigate rosette symmetry. This approach could also give rise to a new kind of puzzle: given a fold pattern on an unfolded piece of paper, and one half-heart to be cut, can a student predict how many total hearts will be cut out?

These types of questions align strongly with the following common core standards:
CCSS.MATH.CONTENT.4.OA.C.5. Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.

CCSS.MATH.CONTENT.5.OA.A.3. Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. For example, given the rule "Add 3" and the starting number 0, and given the rule "Add 6 " and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.

CCSS.MATH.CONTENT.6.EE.A.1.Write and evaluate numerical expressions involving wholenumber exponents.

CCSS.MATH.CONTENT.6.EE.A.2.Write, read, and evaluate expressions in which letters stand for numbers.
a. Write expressions that record operations with numbers and with letters standing for numbers. For example, express the calculation"Subtract y from 5" as $5-y$.
b. Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. For example, describe the expression $2(8+7)$ as a product of two factors; view $(8+7)$ as both a single entity and a sum of two terms.
c. Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). For example, use the formulas $V=s^{3}$ and $A=6 s^{2}$ to find the volume and surface area of a cube with sides of length $s=1 / 2$.

CCSS.MATH.CONTENT.7.EE.B.4. Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.
a. Solve word problems leading to equations of the form $p x+q=r$ and $p(x+q)=r$, where $p, q$, and $r$ are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. For example, the perimeter of a rectangle is 54 cm . Its length is 6 cm . What is its width?
b. Solve word problems leading to inequalities of the form $p x+q>r$ or $p x+$ $q<r$, where $p, q$, and $r$ are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem. For example: As a salesperson, you are paid $\$ 50$ per week plus $\$ 3$ per sale. This week you want your pay to be at least \$100. Write an inequality for the number of sales you need to make, and describe the solutions.

### 3.2.5 Grade 8 and High School Geometry: Isometries

In 8th grade and in high school geometry, the common core standards emphasize the more intuitive isometries of the plane: translation, rotation, and reflection. While the common core does not explicitly mention glide reflections, they are frequently included in high school geometry curricula, and the following activity is significantly richer if you include glide reflections in your discussion.

Older students are ready to name the rigid motions that take one half-heart to another. We present the exercise on the following page as one example of a way to convert a "Cutting out Hearts" Puzzle to a symmetry identification exercise. However, most of the "Cutting out Hearts" Puzzles can be used in this way.

## Quick Answer Key for Exercise on Next Page:

- Half-hearts 1,3 , and 4 are reflections of the original half-heart, and just one fold is required to overlay the original half-heart on any one of them.
- Half-heart 2 is a translation of the original half-heart, and two folds are required to overlay the original half-heart on it.
- Half-hearts 5 and 7 are rotations of the original half-heart, and two folds are required to overlay the original half-heart on either of them.
- Half-heart 6 is a glide reflection of the original half-heart, and three folds are required to overlay the original half-heart on it.


What kinds of isometries take the original half heart to the other half hearts? Translation? Rotation? Reflection? Glide Reflection? How many folds are required to overlay the original half heart onto each of the other half hearts (possibly with other paper between them)?

| Transformation | lsometry | Number of Folds |
| :--- | :--- | :--- |
| original $\rightarrow 1$ | reflection | 1 |
| original $\rightarrow 2$ |  |  |
| original $\rightarrow 3$ |  |  |
| original $\rightarrow 4$ |  |  |
| original $\rightarrow 5$ |  |  |
| original $\rightarrow 6$ |  |  |
| original $\rightarrow 7$ |  |  |

This exercise aligns strongly with the following common core standards:
CCSS.MATH.CONTENT.8.G.A.1. Verify experimentally the properties of rotations, reflections, and translations:
a. Lines are taken to lines, and line segments to line segments of the same length.
b. Angles are taken to angles of the same measure.
c. Parallel lines are taken to parallel lines.

CCSS.MATH.CONTENT.8.G.A.2. Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

CCSS.MATH.CONTENT.HSG.CO.A.5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

CCSS.MATH.CONTENT.HSG.CO.B.6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

CCSS.MATH.CONTENT.HSG.CO.D.12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

## 4 Educational Objectives

"Cutting out Hearts" meets a range of educational objectives, from lofty ideals - joy, curiosity, exploration, perseverance, creativity - to geometric building blocks - reflection, isometry, congruence, construction - to visual-spatial and fine-motor skill development.

### 4.1 Higher Order Goals

Overcoming Challenges. Students deserve genuine challenges and the time and space they need to solve them. "Cutting out Hearts" is one way to do that, in a highly differentiated way that meets each student at their point of challenge.

Focus and Perseverance. Students deserve to learn from experience rather than lecture whenever possible. A teacher can introduce "Cutting out Hearts" in a couple minutes, and then students can immediately dive in, get hooked on the puzzle aspect, and work in a selfdirected way for the rest of class. Math is more fun when you're figuring it out for yourself rather than listening to a teacher explain!

Mathematical Questioning. "Cutting out Hearts" shows students how they can find deep mathematics in unexpected places. It starts with a simple kindergarten activity and grows that activity into something wild and surprising.

### 4.2 Inclusion

Inviting Marginalized Students into Mathematics. "Cutting out Hearts" shares a key positive trait with much of recreational math: it creates space where students who don't love calculation-based math can experience joy and triumph. Students with learning differences and students from groups historically marginalized in mathematics particularly deserve opportunities like this to hear, "This is mathematics too. It is fun, and you are good at it." Frequently the students who solve the most difficult heart-cutting puzzles are not the strongest algebraic manipulators or the math contest stars - and when they feel satisfaction with their puzzle solutions, they also feel more welcome in mathematics.

Connecting Mathematics with Girl Iconography. In US children's marketing, hearts are pink, sweet, frilly, and rhinestone-studded - Barbie, not Oppenheimer. "Cutting out Hearts" sends a message: girl-associated activities lead to tough, deep, mathematical questions.

### 4.3 Cognitive and Motor Skills

"Cutting out Hearts" requires students to exercise both Fine Motor Skills and VisualSpatial Skills. These have repeatedly been associated with mathematical ability in students. While the causative mechanism is a subject of ongoing research, it is likely valuable to work on these skills in a mathematical context. Additionally the pleasure of craft-like work helps many students open themselves to daunting puzzles.

### 4.4 Common Core Standards

### 4.4.1 Standards for Mathematical Practice

As a sequence of geometric reasoning puzzles, "Cutting out Hearts" is aligned with these standards of mathematical practice:

CCSS.MATH.PRACTICE.MP1. Make sense of problems and persevere in solving them.
CCSS.MATH.PRACTICE.MP2. Reason abstractly and quantitatively.
CCSS.MATH.PRACTICE.MP5. Use appropriate tools strategically.
CCSS.MATH.PRACTICE.MP6. Attend to precision.
CCSS.MATH.PRACTICE.MP7. Look for and make use of structure.
CCSS.MATH.PRACTICE.MP8. Look for and express regularity in repeated reasoning.
While the activity does not require students to collaborate, the author certainly encourages it, and nearly all the students she has worked with have chosen to work together part of the time. When students collaborate, the activity is also aligned with

CCSS.MATH.PRACTICE.MP3. Construct viable arguments and critique the reasoning of others.

### 4.4.2 Standards for Mathematical Content

"Cutting out Hearts" aligns naturally with some content standards. In other cases extensions of "Cutting out Hearts" can be used to introduce or reinforce content standards that, on first glance don't seem as clearly connected to the activity. See Section 3.2 for these extension activities. We have listed all the relevant content standards in the following table.

| Immediate/Natural Connection | Connects via Extension Exercise |
| :--- | :--- |
| CCSS.MATH.CONTENT.4.G.A.3 | CCSS.MATH.CONTENT.K.G.A.1 |
| CCSS.MATH.CONTENT.8.G.A.2 | CCSS.MATH.CONTENT.K.G.A.3 |
| CCSS.MATH.CONTENT.HSG.CO.A.5 | CCSS.MATH.CONTENT.1.G.A.3 |
| CCSS.MATH.CONTENT.HSG.CO.B.6 | CCSS.MATH.CONTENT.2.G.A.2 |
| CCSS.MATH.CONTENT.HSG.CO.D.12 | CCSS.MATH.CONTENT.2.G.A.3 |
|  | CCSS.MATH.CONTENT.3.G.A.2 |
|  | CCSS.MATH.CONTENT.4.MD.C.5 |
|  | CCSS.MATH.CONTENT.4.MD.C.6 |
|  | CCSS.MATH.CONTENT.4.G.A.1 |
|  | CCSS.MATH.CONTENT.4.G.A.2 |
|  | CCSS.MATH.CONTENT.4.G.A.3 |
|  | CCSS.MATH.CONTENT.4.OA.C.5 |
|  | CCSS.MATH.CONTENT.5.OA.A.3 |
|  | CCSS.MATH.CONTENT.6.EE.A.1 |
|  | CCSS.MATH.CONTENT.6.EE.A.2 |
|  | CCSS.MATH.CONTENT.7.EE.B.4 |
|  | CCSS.MATH.CONTENT.8.G.A.1 |
|  | CCSS.MATH.CONTENT.8.G.A.2 |
|  | CCSS.MATH.CONTENT.HSG.CO.A.5 |
|  | CCSS.MATH.CONTENT.HSG.CO.B.6 |
|  | CCSS.MATH.CONTENT.HSG.CO.D. 12 |

For completeness, we list here the content standards that align naturally with "Cutting out Hearts" without any need for extension exercises:

CCSS.MATH.CONTENT.4.G.A.3. Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

CCSS.MATH.CONTENT.8.G.A.2. Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

CCSS.MATH.CONTENT.HSG.CO.A.5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

CCSS.MATH.CONTENT.HSG.CO.B.6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

CCSS.MATH.CONTENT.HSG.CO.D.12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

## 5 Theoretical Considerations

### 5.1 Pedagogical and Theoretical Contrast with Fold-and-One-Cut

"Cutting out Hearts" clearly bears some resemblance to fold-and-one-cut activities. However there are theoretical and pedagogical differences that are worth highlighting.

The Fold-and-One-Cut Theorem roughly states that, given any collection of straight line segments in the plane, there exists a flat folding of the plane that maps these line segments (and nothing else) to a line. This was proved by Demaine, Demaine, and Lubiw in 1998.

Practically, this means that one can cut out any polygon one wishes by folding a piece of paper several (possibly many) times and then cutting along one straight line. This is often implemented in math explorations by giving students printouts of polygons which they then fold and cut out with one single cut.

### 5.1.1 Differences in Implementation and Pedagogical Goals

"Cutting out Hearts" has a different feel from fold-and-one-cut because it requires cutting out many hearts simultaneously, rather than a single polygon. This is consistent with the original motivating problem of cutting out snowflakes ${ }^{2}$, which tend to appear more artistic and complex than the polygons associated with fold-and-one-cut exercises.
"Cutting out Hearts" is implemented so that students begin with blank or old scrap paper, rather than paper with the heart designs printed on it. This pushes students to visualize the folds they will need to make rather than folding thick lines on top of each other; this visualization work may be a higher level of visual spatial skill development. Additionally this approach allows us to use old scrap paper, which is cheaper and less wasteful. While this approach is also sometimes used with fold-and-one-cut activities, it is less common.

### 5.1.2 Differences in Theoretical Aspects

On the theoretical side, the Fold-and-One-Cut Theorem does not apply to curved heart shapes. This may seem like a technicality since curved shapes can be approximated by polygons, but

[^1]in fact the requirement that curved sides be folded onto each other so that they match exactly leads to different constraints, and substantially different types of solutions, as described in the next section.

### 5.2 Fold-and-One-Stamp-Cut

"Cutting out Hearts" is a special case of what we'll call the fold-and-one-stamp-cut problem: Given a finite collection of congruent shapes (where shapes are simple closed curves, like polygons or outlines of hedgehogs) in the plane is it possible to make a flat folding of the plane that layers all the shapes exactly on top of each other, with no other parts of the plane included in the layering, so that all the shapes could be cut out simultaneously with a single stamp cut?

Hearts are a special case because they have a line of symmetry. To better illustrate fold-and-one-stamp-cut problems, we might create designs from repeated outlines of hedgehogs (which have no nontrivial symmetry), and ask solvers to fold paper so that the hedgehog outlines agree. However, pedagogically, hearts have many advantages: familiarity, association with girls' activities, ease of cutting with scissors.

### 5.3 Conjecture

Spoiler Alert: do not read this section or look at the pictures on the next page if you don't want to know how to solve puzzle W.

We conjecture that, for any finite collection of non-intersecting congruent bounded shapes in the plane, it is possible to uniformly shrink all of the shapes (say, by dilation around their centers of mass) enough that there exists a flat folding of the plane that maps all the shapes and no other points to a single copy of the shape, so that they can all be cut out with one punch of a stamp cutter, without cutting any other part of the plane.

In the following section, we will prove this for mirror-symmetric shapes like hearts.
However we are not confident about this conjecture in the more general case of shapes without reflection symmetry. Nonetheless we offer a demonstration that it is true for a collection of just two shapes. This demonstration is essentially due to Jonah H, who, as an 8th grader, was the first person to solve Puzzle W using this method.

Consider two congruent shapes $H_{1}$ and $H_{2}$ inside a planar region $R$. Since we are allowed, by the terms of the conjecture, to shrink the shapes as much as we like, we can make the distance between them as large as we like compared to their size. Let $S$ be a long narrow ribbon-like strip containing the two shapes, with one shape at either end of the strip. It is possible to accordion-pleat $R$ so that only $S$ is visible. Then $S$ can be folded several times to perfectly layer $H_{2}$ on top of $H_{1}$. (See Figure 8 for the example of a horizontally translated shape with no nontrivial symmetry.)


Figure 8: Solving Puzzle W, except with hedgehogs. More generally, overlaying any two distant congruent shapes.

It is worth noting that in the case of shapes with no reflection symmetry, an infinite set of folds is likely needed, as in the case of the pleats in the above example.

By contrast, in the case of mirror-symmetric shapes, it suffices to use a finite set of folds which rotate the shapes in the manner of origami twist folds. See Figure 9 for an example of twist folds that begin to convert Puzzle X into the more easily solvable Puzzle M, albeit with smaller hearts.


Figure 9: Using triangle twists (a technique from origami tessellations) to change the orientation of hearts. Mountain folds are magenta dash-dot lines, and valley folds are turquoise dashed lines.

Mirror-symmetric shapes and shapes without reflection symmetry differ in a critical manner: mirror-symmetric shapes can be folded on top of each other with the same side of the paper touching. By contrast, shapes without reflection symmetry must be overlaid with different sides of the paper touching, as demonstrated in Figure 10. This may lead to unresolvable paper collisions, causing our conjecture to fail for shapes without mirror symmetry.


Figure 10: Hearts and hedgehogs on paper that is orange on one side and yellow on the other. The hearts, which are mirror-symmetric, can be overlaid with the yellow sides touching. By contrast, the hedgehogs, which do not have reflection symmetry, must be overlaid so that the orange side of one hedgehog touches the yellow side of the other hedgehog.

### 5.3.1 A Proof in the Case of Mirror-Symmetric Shapes

In fact our conjecture is true in the case of mirror-symmetric shapes and stamp cutters that cut out half of each shape. That is, all heart puzzles are technically solvable if we are allowed to shrink the hearts to an arbitrarily small size.

The proof is by construction; we will present an algorithm in four steps, with possible shrinking occurring at each step. In each of the figures that follow, dashed lines indicate valley folds and dot-dashed lines indicate mountain folds.

Step 1: Rotate all the shapes, so that their symmetry axes are all distinct and oriented in the same direction. We can rotate a single shape using origami machinery called an open-back square twist, as shown in Figure 11. Such twists can rotate any square region up to $90^{\circ}$ relative to the rest of the plain.

We choose a direction $v$ such that no two shapes have centers of mass that lie on a line with direction $v$; this avoids a circumstance in which two shapes share the same symmetry axis. We then rotate each shape serially, using open-back square twists, so that each shape is oriented with its symmetry axis aligned with $v$, as shown in Figure 12. This may require multiple twists if a shape must be rotated $90^{\circ}$ or more. We are always guaranteed that an open-back square twist rotation by a given angle under $90^{\circ}$ is possible because we can choose any two perpendicular directions for the pleat folds, and only a finite number of possible choices create conflicts where a mountain crease of the pleat folds passes through the center of mass of another shape. (In some cases the mountain creases will come too close to other shapes, but this is avoided by shrinking the shapes further and using an open-back square twist with a smaller square size.)


Figure 11: An open-back square twist. The outer square will be twisted by an angle $2 \theta<90^{\circ}$ relative to the rest of the paper, and the gray shaded square region will not overlap any other paper layers after folding. (It is possible to twist the square by exactly $90^{\circ}$, but this results in multiple layers of paper everywhere in the interior of the square, which interferes with the fold-and-stamp-cut process.)


Figure 12: Twisting all shapes so that they have the same orientation. The three shapes on the left have already been twisted so that their symmetry axes are vertical, with all hearts right-side-up. The middle shape is shown being twisted, and the three shapes on the right will be twisted next.

Step 2: Compress the shapes into a long narrow rectangular region. Without loss of generality, we rotate the paper so that the symmetry axes of the shapes are all vertical. We then use horizontal pleats, as in Figure 13, to move the shapes into a long, thin rectangular region, so that any two shapes have greater horizontal separation than vertical separation. As before, if any two shapes are too close together to make such pleats, we shrink the shapes until the pleating is possible.


Figure 13: Translating all shapes vertically by means of horizontal pleats so that they fit in a rectangle that is wide horizontally and narrow vertically. The shapes have been shrunk so that the pleats can be made wide enough that the horizontal separation between any two shapes is greater than their vertical separation.

Step 3: Align all shapes horizontally. We use $45^{\circ}$ diagonal pleats, as in Figure 14, to align all shapes horizontally. The existence of such a pleating is guaranteed by our prior choice to ensure that the horizontal separation between any two shapes is greater than their vertical separation. As usual, if the pleating overlaps any shapes, we shrink the shapes until the overlap no longer occurs.


Figure 14: Translating all shapes diagonally by means of $45^{\circ}$ pleats so that the shapes are horizontally aligned.

Step 4: Overlay all half-hearts. Finally, we pleat the plane, by means of mountain folds along the shapes' symmetry axes and valley folds midway between successive symmetry axes, so that all half-shapes are exactly overlaid, as in Figure 15. This completes the proof.


Figure 15: Using vertical pleats to overlay all half-hearts.

### 5.4 Impossible Configurations

We suspect that many collections of congruent shapes are impossible. For example, in the limiting case below of two tangent hearts shown in Figure 16, it seems necessary to fold along the tangent line to have any hope of overlaying the hearts - and since the tangent line is not a line of reflection, the two hearts will not lie exactly on top of each other.


Figure 16: Two tangent hearts. These cannot be folded exactly on top of each other.

More generally, it may be that hearts that take up too much of the paper in a non-mirrorsymmetric configuration, like those in Figure 17, are impossible. Deciding if a heart puzzle is possible without shrinking the hearts is a significant topic worth further study.


Figure 17: It seems unlikely that these hearts can be folded on top of each other without first shrinking the hearts.

### 5.5 A Technical Point

Fold-and-one-cut algorithms work in part because they are allowed to pleat straight line segments on top of themselves many times. In fact this can be done with many shapes whose boundaries are composed of segments of constant curvature. In the puzzles above, we have purposely chosen a heart shape whose boundary has no segments with constant curvature to prevent this approach. It is possible that the hearts in Figure 18, with appropriate trimming of the boundary region, can be folded so that the two hearts, whose boundaries are composed of straight line segments and circular arcs, can be cut out with a single cut composed of a short straight segment and a short circular arc.


Figure 18: These hearts are composed of straight line segments and circular arcs. Consequently they admit many pleated folds along their constant curvature boundary segments.

## 6 References

Adams, S. (2012). Origami Tessellation Basics: Open-Back Triangle Twist [Video]. Happy Folding YouTube Channel. https://www.youtube.com/watch?v=UKdhEcvgWKQ

Atit, K., Power, J. R., Veurink, N., Uttal, D. H., Sorby, S., Panther, G., Msall, C., Fiorella, L., \& Carr, M. (2020). Examining the role of spatial skills and mathematics motivation on middle school mathematics achievement. International Journal of STEM Education, 7.

Batterson, J., Rogers, S., \& Guillet, K. (2016). Beast Academy: Practice 5C. AoPS Incorporated.

Demaine, E. D., Demaine, M. L., \& Lubiw, A. (1998) Folding and one straight cut suffice. Technical Report B-98-18, University of Waterloo.

Demaine, E. D., \& O'Rourke, J. (2007). Geometric Folding Algorithms: Linkages, Origami, Polyhedra. Cambridge University Press.

Hart, V. (2012). Snowflakes, Starflakes, and Swirlflakes [Video].
https://www.youtube.com/watch?v=8EmhGOQ-DNQ
Hart, V. (2017). Hexaflexaflakes [Video]. https://www.youtube.com/watch?v=DIyruYQ-N4Q
Lang, R. J. (2014). Twists, Tilings, and Tessellations: Mathematical Methods for Geometric Origami. CRC Press.

National Governors Association Center for Best Practices \& Council of Chief State School Officers. (2010) Common Core State Standards for Mathematics. Washington, DC.

Pitchford, N. J., Papini, C., Outhwaite, L. A., \& Gulliford, A. (2016) Fine Motor Skills Predict Maths Ability Better than They Predict Reading Ability in the Early Primary School Years. Frontiers in Psychology, 7.

Primary Mathematics (Standards Edition): Textbook 1B, Workbooks 1B and 2B [popularly known as Singapore Math]. (2008). Marshall Cavendish.

Yoder, M. (2023, December). Advent of Tess [Video and origami pattern series]. Gathering Folds. https://training.gatheringfolds.com/adventoftess23

Young, C. J., Levine, S. C., \& Mix, K. S. (2018) The Connection Between Spatial and Mathematical Ability Across Development. Frontiers in Psychology, 9.


[^0]:    ${ }^{1}$ As shown in Section 5.3.1, all puzzles of this type are solvable if we are allowed to shrink the hearts to an arbitrarily small size. However we do not know of solutions for $\mathrm{X}, \mathrm{Y}$, and Z that preserve heart size. In addition, we do not know if solutions that only use simple folds (straight folds through the entire paper, rather than twist folds like those in Figure 11) exist, even if we are allowed to shrink the hearts.

[^1]:    ${ }^{2}$ The author tried and failed to teach her kindergarteners to make paper snowflakes. They loved folding and cutting paper, but couldn't wrap their heads around the diagonal folds required for rosette symmetry. Instead they made rectilinear folds and were delighted by the funny patterns they made. She decided to reverse the process and challenge them to create these rectilinear patterns purposefully. Since Valentine's Day follows winter snowflakes, hearts were the obvious next step. Once the kindergarteners were on a roll cutting out grids of hearts, she took the activity to her first and second graders, adding some slightly harder puzzles. Eventually she stretched the puzzle range all the way up to the eighth graders.

