NATIONAL MUSEUM OF MATHEMATICS

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How is a Polygon Like a Circle?

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How is a Polygon Like a Circle?

Contents

Introduction

In this arts-integrated activity students walk along the edges of a polygon marked on the ground or floor and turn through its internal angles, keeping track of the sum of those angles. This allows them to establish a formula for the sum of the internal angles based on the number of the polygon's vertices. They also walk and sum the external angles. These movement patterns are then combined with a playful and creative movement arts activity. Additional activities make connections with the rotations of heavenly bodies, puzzles with coins and circles, circular movements in dance, and the concepts of turning and winding number.

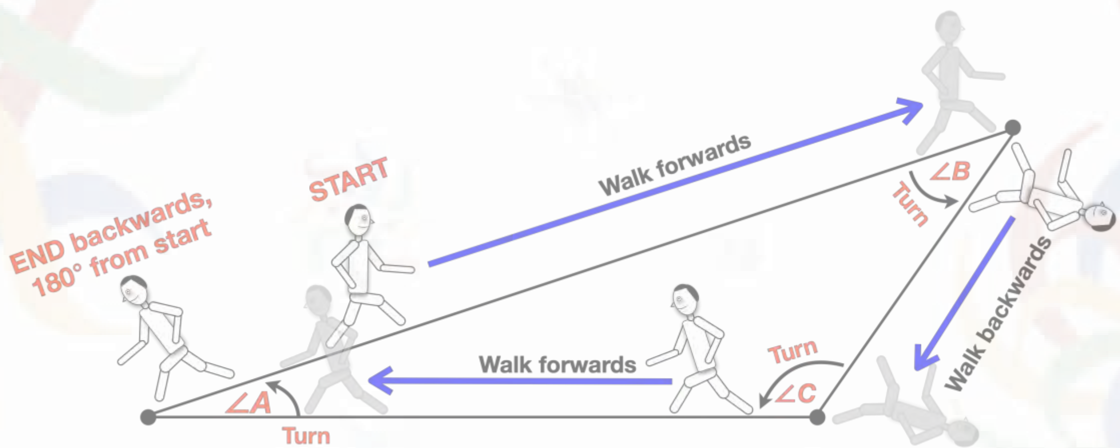


Figure 1. *Walking the internal angles of the triangle demonstrates that the sum of those angles is 180°. At the end the student is standing backwards from the starting position, so must have executed an odd number of half-circles.*

Outline

Note these may be accomplished in longer segments over two or more days. Here we suggest a total of two sessions, 45 minutes each, over two days.

Preliminary Activities
We find that many teachers prefer to precede the movement activities described here by doing a preliminary review of angle measure with their students, possibly using the Angle Measure Review handout in the Appendix. Some tell us they like to do an activity such as described in the Appendix handouts using either origami or tessellations to find the sum of the internal angles of triangles and quadrilaterals.

Day one

1. 10 minutes. Preliminary practice standing and turning to right and left by multiples of 90°, and finding angle sum of those turns.
2. 5 minutes. Have several students demonstrate for the class how to walk the exterior and internal angles of a non-square rectangle, with one student in the center acting as a "human adding machine," as described in the Walking Angle Sums section and in the student handout.
3. 25 minutes. Walking the sum of both the external and internal angles of a triangle, quadrilateral, and pentagon. Either use the two pages of the student handout below,

or possibly replace the second page of the handout with a different set of simple polygons.

4. 5 minutes. Discussion and reflection on the activities and findings from day one. The goal is for students to either conjecture or perhaps give initial explanations for the formula $(n-2)180°$ as the sum of the internal angles of a simple polygon with n vertices.

Day two

5. 10 minutes. Further discussion of day 1 activities and findings. Allow students to report the findings of their groups, perhaps displaying diagrams on the board.
6. 20 minutes. Creation and performance of movement compositions using polygonal paths.
7. 5 minutes. Small group and class discussion on student compositions and discussion of relation to polygonal angle sum activities. For example, did the students keep track of their rotations as they moved around the polygons?
8. 10 minutes. Pose additional problems using several more complex polygons, for example one of the other more complex examples shown in Figure 3 or one of the star polygons shown in the Polygon Vocabulary section.

Additional Activities

9. 30 minutes. Discussion of patterns of circling motion of heavenly bodies such as planets, moons, and asteroids, using the handout in the Appendix. Physical activities exploring the effects of gravity.
10. 30 minutes. Physical activity learning about and practicing circular dance movements. See The Circles in Dance section in the Appendix for suggestions.
11. 30 minutes. Circular puzzles with small circular objects such as quarters, using the handout in the Appendix.

Materials

1. Handouts describing activities.
2. Material for marking non-permanent polygonal shapes on the ground or floor: sidewalk chalk, painters' tape, or rope loops.
3. Open area without obstructions for polygonal paths on the ground or floor and for movement activities. If possible, desks in a traditional classroom might be moved aside to create such a space.
4. Music and speaker for accompanying creative movement activities.
5. Scissors for cutting out paper polygons, if adopted as a preliminary activity.
6. A whiteboard or blackboard or large sheets of paper on which to display students' diagrams and calculation results.

Movement Activities in the Math Class

Some mathematics teachers will be more comfortable with overseeing movement activities than others, so one approach would be to co-teach the workshop with a dance or physical education teacher. However, the learning curve for implementing integrated movement and math explorations is not great, especially if the movement demands, as in this activity, involve only ordinary actions like pedestrian walking forwards and backwards, and variations on such movements in the final movement composition task.

A checklist is included in this document as an aid to assessing student movement composition work. Perusing the checklist will give readers who may be unfamiliar with leading math and movement activities ideas about what to look for and encourage students to work towards in creating movement sequences that connect with their mathematical work. The primary mathematical activity has students walking around the polygons in a very orderly and systematic way. However, when they start to compose movement sequences they will easily allow themselves to be creative with that process.

Students may create sequences that colorfully communicate the mathematics, or they may enjoyably make dance movement that communicates something else entirely. Support their creative impulses! But if their work is primarily about the mathematics urge them to take creative and artistic license with how they communicate those concepts. And if their work around the polygons goes in other than mathematical directions, ask them to reflect on and perhaps incorporate some of the mathematical elements. Three entertaining examples students have created have had students cleverly crisscrossing the edges of star polygons in exciting ways; seeming to chase each other around polygon edges; and streaming around the edges in opposite directions without bumping into each other.

Goals

Students directly turn through the internal and external angles of a polygon marked on the floor or outdoors on the ground, and in the process develop the formula for the sum of those angles. They then create fun movement patterns that play with the circling movements involved in following polygonal floor paths. The activity thus empowers students to employ embodiment of mathematical concepts to further their understanding at the same time it gives them experience developing creative movement skills. Additional warm-up activities give students practice in doing turning movements. Other warmup activities engage students in smaller scale kinesthetic activities using manipulatives to demonstrate the sum of polygonal angles. Further activities explore the use of turning in movement arts, make connections to the orbital patterns of heavenly bodies such as planets, moons, and asteroids, or engage the students in solving puzzles about circular motion.

(a) (b) (c)

Figure 2. *(a) Painter's tape, (b) Sidewalk chalk, (c) Rope loop and garden stakes*

Prepare the Polygons

We have used several methods for creating polygons on the ground. Each has advantages and disadvantages! Although page 2 of the student handout on pages 7 and 8 shows three each of triangles, quadrilaterals, and pentagons, you may not have space for this many; or you may have enough students that you need more. The nine shown should be enough for a class with 20 to 30 students working in groups of threes or fours. But you may decide to replace page 2 of the handout with your own polygons. A good diameter for each polygon is 7 or 8 feet.

1. In a classroom use 1 inch thick painter's tape. A 60 yard roll usually costs about $5. One roll is usually enough for polygons for one class session. Use of this in the classroom will require that it is easy to move desks and chairs aside to create an open space. Be aware that painter's tape comes up easily if placed on the floor for a short time, but the longer it remains and is stepped on the more securely it will stick to the surface!
2. Outdoors on blacktop use sidewalk chalk. This is extra thick chalk that is often a little more crumbly than typical blackboard chalk. A box of 15 to 20 of this extra thick chalk in a variety of colors should cost $6 to $8 and should be enough for one class session. Chalk takes some effort In bending over to draw the polygons. I've drawn about 12 to 15 polygons in about 30 minutes. If it does not rain and the area is not walked on much the polygons may remain for several days.
3. Outdoors on dirt or grass use rope loops. I have used ½ inch thick nylon braided rope, each loop at least made from about 13 feet of rope. I use cloth gaffer's tape to secure the ends though ordinary plastic duct tape can work too. I have used "ground staples" or garden stakes for holding the rope loops in place on the ground, though I made the mistake of using 12 inch long staples; 6 inch staples are easier to push into the ground! About 50 such stakes cost about $6, and it will take about 50 stakes for a first class. The braided nylon rope costs about $15 for 100 ft. of ¼ inch or $25 for 50 ft. of ½ inch rope. You might need a total of 100 ft. for a first class session. We use the loops for other math and movement activities, so the extra expense has made sense to us. I have used a rubber mallet to pound in the garden stakes. One might not use the stakes, but then the rope will tend to move around, especially during the creative movement section.
4. A fourth possibility, used in [6], is for students to hold up the rope, one student per polygon vertex. Unfortunately this does not work for the star polygons, since

students would get tangled in the rope crossings, and it can be difficult for the students to maintain a static polygon.

In all cases you will want each polygon to be at least six or seven feet in diameter. The polygons with more sides should be a bit larger than the triangles and quadrilaterals. The next page is a handout showing one possible selection of polygons. You might want to have more of the simpler polygons to start out, that is more triangles and quadrilaterals, leaving the star polygons for later.

Figure 3 shows a selection of more complex polygons. See the Further Discussion and Background section for more details on these. Figures 3(a) and (b) are simply concave and should not give students much trouble. Figure 3(c), a “self-intersecting” polygon, has 11 vertices. Its “winding number” around vertex P, as explained in the Polygon Vocabulary section, is 2, since a walker turning through external angles while walking clockwise completes 2 complete turns around P in the clockwise direction. However, the turning number, the number of times the external angle walker completes circles around their own center is 4. This implies the angle sum for this 11-vertex self-intersecting polygon should be $(11 - 4 \cdot 2)180° = 3 \cdot 180°$, or three half-circles, which it is.

Figure 3(d) changes direction as the walkers move around it, and it can generate discussion about which are the external and which the internal angles (reasonable mathematicians may disagree!) – but it is therefore good for extra exploration. It is known as a “crossed polygon.” Figure 3(e), the {8/2} star polygon, has two disjoint sections, which takes two walkers, but satisfies the angle sum formulas. It is known as a “compound polygon.” Note that figure 3(f) is a connected set of three triangles which has internal angle sum of $3 \cdot 180°$, though it has 7 vertices. Figure 3(g) has 8 vertices, and if we divide the interior into triangles with all their vertices the vertices of the figure, as in the figure at the bottom of the Summing the Angles with Origami page in the Appendix, we see that the sum of the interior angles is $8 \cdot 180°$. Such examples can provide further challenges for students. For even more extreme polygon examples, see “Are Your Polyhedra the Same as My Polyhedra?” by Branko Grünbaum [4].

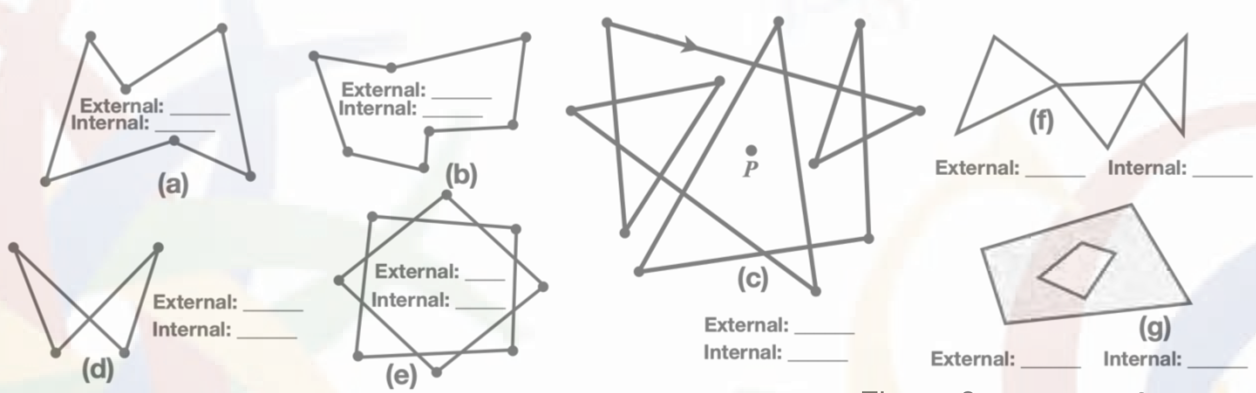
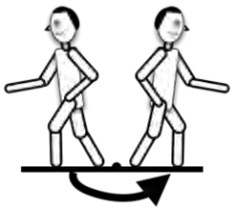


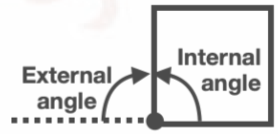
Figure 3. *A variety of more challenging polygons*

How is a Polygon Like a Circle?

Names of group members:____

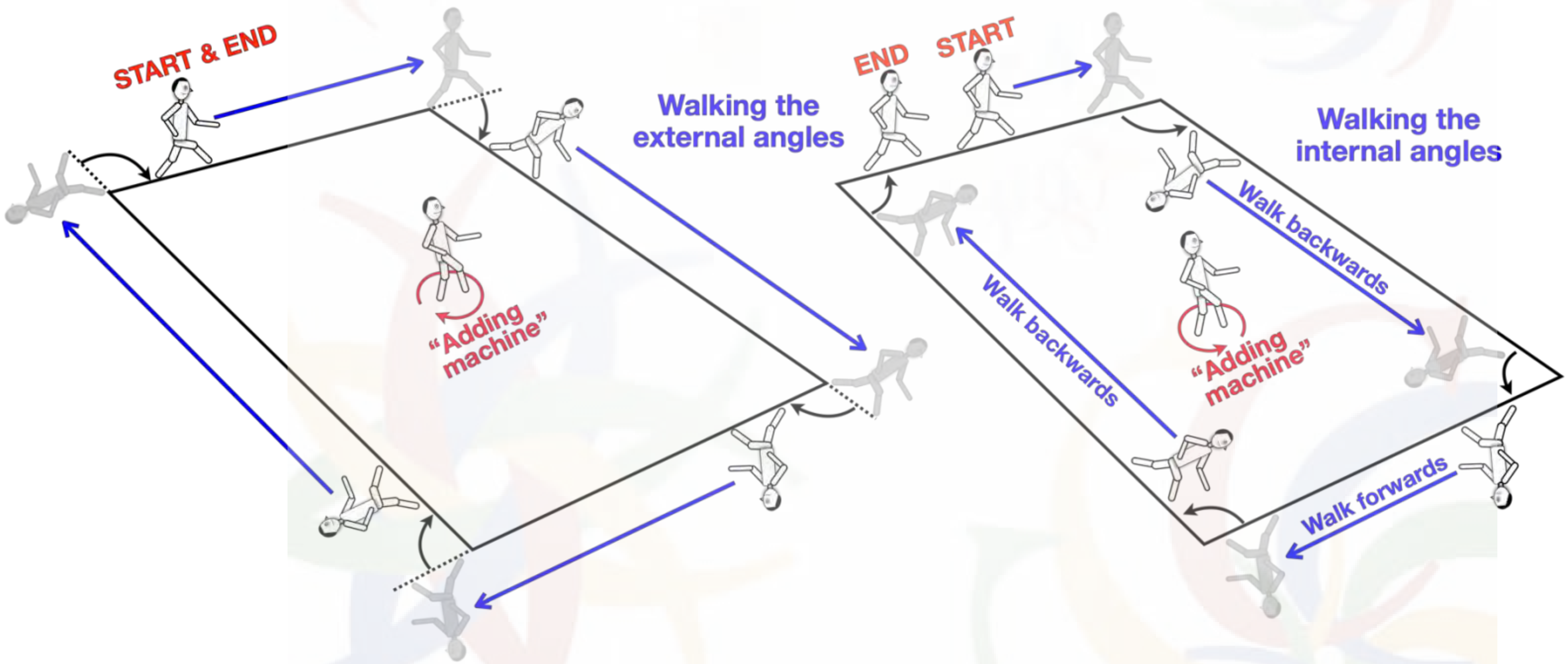


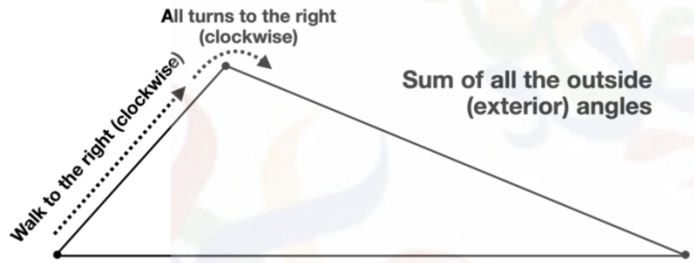
This is a half-turn: We call it a **180 degree** turn.

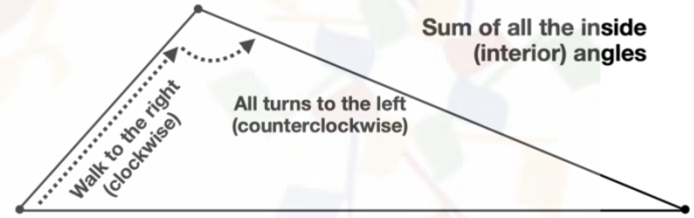


We are going to add up the **inside** or **internal angles** of a number of polygons and see if they add up to a whole number of half-turns. We'll do the same thing with the **outside** or **external angles** of polygons.

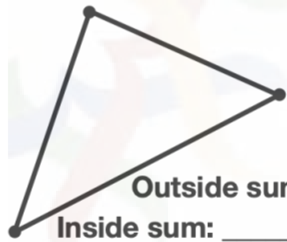
Work in groups of three or four. One person walks clockwise around the polygon and turns clockwise, thus "adding up" the outside or external angles. Another person walks clockwise around the polygon but turns counterclockwise "adding up" the inside or internal angles. A third keeps track of the total amount of turning by standing in the center and acting as a human "adding machine," duplicating the walkers' turns. And one person writes down the **total number** of **half-circles** in the sum of all the external or outside angles and also all the internal or inside angles. Work on at least one triangle, one quadrilateral, and one pentagon.



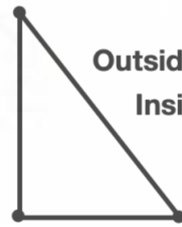




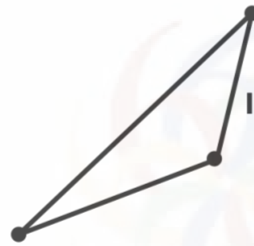
A
Triangles



Outside sum: ______
Inside sum: ______



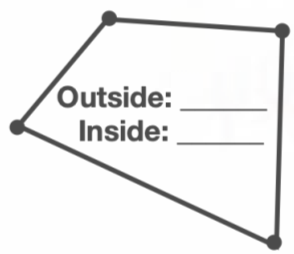
Outside sum: ______
Inside sum: ______



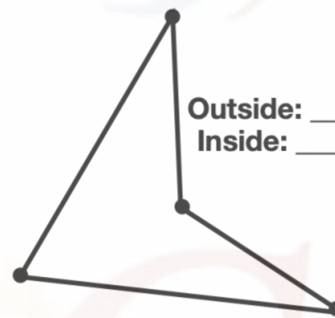
Outside sum: ______
Inside sum: ______

B
Quadrilaterals

Outside: ______ Inside: ______

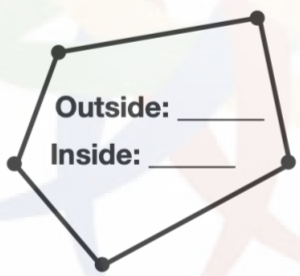


Outside: ______
Inside: ______

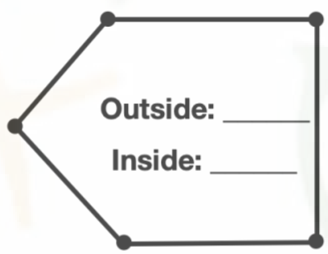


Outside: ______
Inside: ______

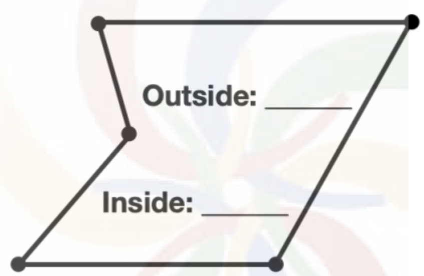
C
Pentagons



Outside: ______
Inside: ______



Outside: ______
Inside: ______



Outside: ______
Inside: ______

Warmup with Angle Jumps

The following involves small jumps, which may be easy for most students. But a bit of a prior warmup is always helpful to avoid injuries: turn on some light instrumental music and lead students in simple bending of the knees (plies), reaching the arms upward while raising the weight onto the tiptoes, shaking one leg and then the other, one arm and then the other. Do gentle and slow twisting movements of the torso. Make large circles with the arms. The goal is to increase circulation, warm up musculature, and prepare the body for more dramatic movements.

Because students often have trouble determining whether to turn to their right or left as they walk the angles of the polygons, we do a physical warmup that focuses on fun and easy turning movements. We alternately use the terms “turn clockwise” and “turn right” to familiarize students with that vocabulary; both indicate a turn in the clockwise direction from the point of view of an observer above the class. Or else “counterclockwise” and “to the left” for those turns. We mostly use turns of 90° or 180°, or “quarter turn” and “half turn”, since they are easily executed by most students.

The teacher leading the class shouts a command and claps hands and students must execute that turn immediately. After a short sequence of such turns the teacher stops and asks the students how much of a circle in total was just executed by the class. A local teaching artist uses and suggested this version of the exercise: the teacher says “right” (or “clockwise”), and the students turn their heads to look right, followed by “90 degrees” and a clap and the students do a standing jump from two feet to two feet to the right of 90 degrees. For example:

“Left”… “quarter turn” … clap
“Right”…”180 degrees” … clap
“Right” … “90 degrees” … clap
“Clockwise” … “half turn” … clap
“Counterclockwise”… “quarter turn” … clap
“Right” … “180 degrees” … clap
“Clockwise” … “quarter turn” … clap
“How much of a circle is the total?”

Students should answer one and a half circles to the right or clockwise. You may also ask them to say how many total half circles resulted from that sequence. Many variations are possible. If students are used to using movement in classes, perhaps stop leading and ask a student to continue to lead this warmup. Turn on music that has energy but is instrumental and is not suggestive of popular styles of dance movement – and has a tempo that is not too quick or too slow. Best are medium energy works that establish a supportive musical environment without an overbearing beat and without lyrics.

When students are warm try variations such as shouting “270 degrees” or even “full circle.” Or try “one third turn” or “120 degrees” or some multiple of 45 degrees. Try “walk around three half-turns very slowly.” Be inventive!

Walking Angle Sums

`Demonstrate for students how to walk and turn through the external angles and the internal angles of a simple polygon. A good choice might be a non-square rectangle, since then all the turning angles are easily identifiable as 90°. (Note we are ignoring the distinction between positive and negative angle measure; see the Polygon Vocabulary section for another note about this.) Have one student serve as the “adding machine,” responsible for accumulating the turns of the walker. The adding machine person starts facing the same way as the walker and turns in the same direction and the same amount as the walker. We recommend that both walkers traverse the polygons in the clockwise direction so that external angles will all be turned clockwise and internal angles counterclockwise (reasons for doing this is discussed in the Further Discussion section.)

Suzannah Young, an eighth grade math teacher at the Watsonville Charter School of the Arts, recommends to students that they carefully walk with one foot along the polygon edge, right foot for the external angle walker, left foot for the internal angle walker. The teacher also recommends using “turn” to describe individual turns at the polygon angles and “circling” or “circles” to describe the sum of the angles, since students often become confused about whether we are counting the number of times they turned or the total sum of those turning angles. It is probably best to have students start and end in the middle of an edge rather than at a vertex, since otherwise they may neglect to turn the final angle. The same teacher also has students notice that when they turn through an internal angle they are equivalently turning through the opposite “vertical angle,” thus supporting a kinesthetic understanding of that geometric concept (see the diagram in the Polygon Vocabulary section).

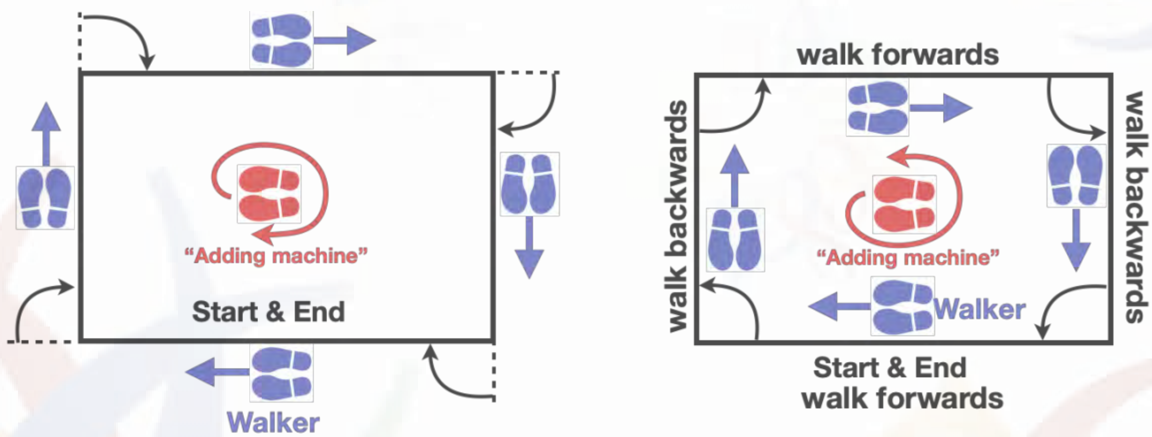


Figure 4. *On the left the walker turns through the external angles, on the right the internal angles. In both cases the sum is one complete circle, or two half circles for the rectangle.*

Polygon Movement Composition

After students have tabulated the numbers of half-circles in the sums of external and internal angles, and discussed patterns that they see, have them work in groups to create a movement composition around one or more polygons. This is an open-ended assignment, but adding constraints may make it easier for them to complete the task:

1. Each student in each 3- or 4-person group should participate in some fashion.
2. Each student should traverse the polygon path at least once, though direction of traversals may vary.
3. Students should pay attention to how many times they circle around the center of the polygon as well as how many times they circle their own center, and in which direction, and be able to state these counts afterward, as this relates clearly to the earlier mathematical investigation.
4. Their movements should be carefully yet energetically performed. Their group might use uniform movement qualities or might decide that some in the group use contrasting movement styles.
5. Play instrumental music as background while they work and as score for their compositions.,
6. Have the student groups perform for each other and discuss what they see their classmates do. Usually it is best to have all groups to perform in quick succession, with applause held until the end.
7. After they have performed once send them back to rework their compositions, trying to iron out any difficulties, then perform again for the class.

Reflection and Discussion

1. Suzannah Young, mentioned above, has her students record their observations and thoughts about this activity in their math notebooks as a preliminary step to discussion.
2. Have students tabulate their angle sum discoveries in groups and then on the board and discuss as a class. See the angle and half circle data chart for the polygons suggested in this document on the next page.
3. What patterns do students see in the data?
4. Can they generalize and apply to other polygons that they have not tested? Draw such a polygon on the board and ask students to decide how many half-circles make up the sum of the internal angles. What was their reasoning process?
5. What mathematical questions do they still have about this work?
6. Ask students to discuss their experiences creating movement sequences using the polygonal paths.
7. What mathematical elements did they include in their compositions?
8. What artistic ideas did they include in their compositions?
9. How might they want to further develop their compositions?

Half Circle Data for Suggested Polygons

Half circle data for suggested polygons					
Convex polygons	Number of vertices	Winding number around central point	Half circles in the external angle sum	Half circles in the internal angle sum	
Triangle	3	1	2	1	
Quadrilatera	4	1	2	2	
Pentagon	5	1	2	3	
Hexagon	6	1	2	4	
Heptagon	7	1	2	5	
Octagon	8	1	2	6	
Star polygons	Number of vertices	Winding number around a central point	Half circles in the external angle sum	Half circles in the internal angle sum	See the Polygon Vocabulary page for definition of {*n*/*k*}.
{5/2}	5	2	4	1	
{6/2}	6	2	4	2	
{7/2}	7	2	4	3	
{7/3}	7	3	6	1	
{8/2}	8	2	4	4	
{8/3}	8	3	6	2	
Complex polygons	Number of vertices	Turning number	Half circles in the external angle sum	Half circles in the internal angle sum	Notes
Fig. 3d	4	0	0	4	Internal angles are all CCW, so some appear external to the figure! External angles taken as positive and negative.
Fig. 3e	8	2	4	4	Sum angles in the two sections together, as two quadrilaterals.
Fig. 3c	11	4	8	3	The turning number is found by continuously turning CW.

Further Discussion and Background

This is an alternative to a popular demonstration that directs students to turn through the exterior angles while walking along the edges of a polygon on the floor and then do an algebraic calculation that establishes the formula for the sum of the interior angles. Although the algebraic manipulation demonstrates the power of symbolic work, directly counting the sum of the internal angles may provide a more immediate "aha" moment and help students understand how the mathematical concepts are clearly embodied in how we move around in space every day.

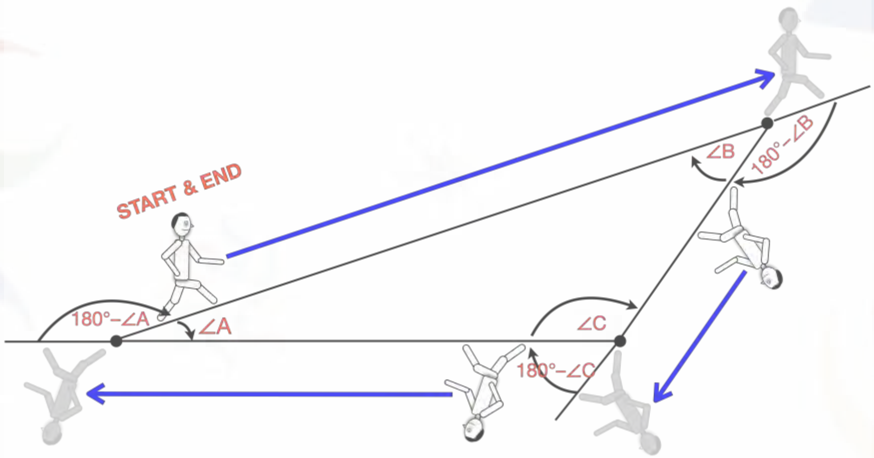


Figure 5. *Walking around the triangle adds the "external angles" of the triangle.*

The popular method of walking the exterior angles is based on the fact that each exterior angle plus the associated internal angle has sum 180°. In the case of a triangle the external angle walker clearly turns a total of 360°, a full circle, around the triangle in order to return to the starting point, and this is equal to the sum of those external angles,

$$360° = (180° - \angle B) + (180° - \angle C) + (180° - \angle A)$$
$$= 3 \cdot 180° - (\angle A + \angle B + \angle C)$$

And with a bit of algebra this reduces to

$$\angle A + \angle B + \angle C = 180°$$

For a convex *n*-sided polygon students may apply this method to find that 180(*n*–2) degrees is the sum of the internal angles.

Many students find that such symbolic algebraic manipulations obscure what is really going on, so in this activity we use the walking demonstration in which the walker turns through the triangle's internal angles. This method nicely extends to finding the sum of the internal angles of polygons with more than 3 sides, as well as to more complex figures such as concave or self-intersecting polygons in Figure 3 or the star polygons shown in the Polygon Vocabulary section.

In our version of this activity students are directed to "walk to the right" or clockwise around the polygons while walking the external angles. They also circle around the polygon to the right when walking the internal angles but turn through each internal angle counterclockwise. These directions are designed to coincide with the traditional mathematical notation in which counterclockwise angle measure is positive and clockwise angle measure is negative. But note that in the Figures in this document we have mostly ignored the distinctions between positive and negative angles so as to simplify the descriptions for students who may not have studied angle measure in any depth. We also use degrees rather than radian measure in which, for example, π radians corresponds to 180°, as radian measure would be unfamiliar to most middle school students. This approach also relates to more advanced concepts of winding and turning numbers, as described in the Polygon Vocabulary section.

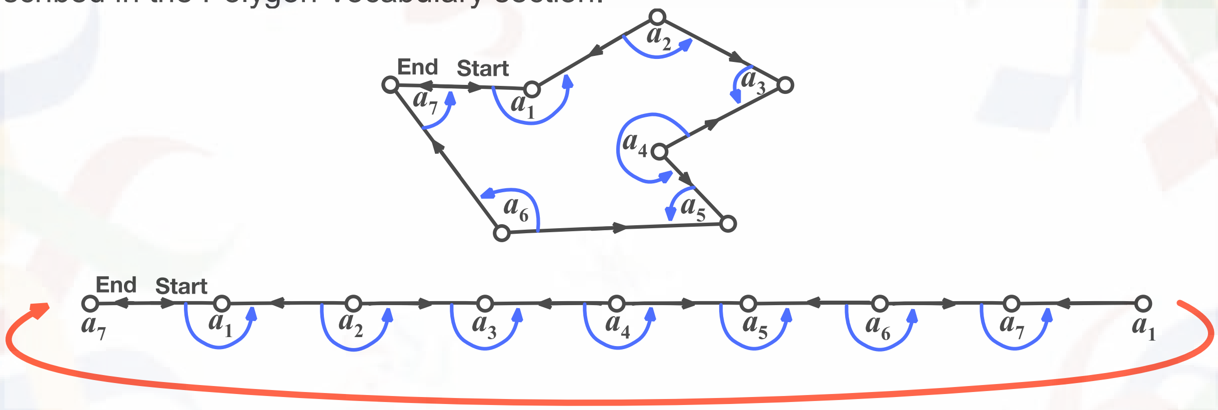


Figure 6. *An easy way to understand the internal angle sum of most polygons.*

On the other hand, delineation of positive and negative angle measure allows for a heuristic or intuitive but non-rigorous shortcut for understanding the formulas for total internal angle measure. In the top figure above suppose a student walks forward along the first edge, turns through the internal angle a_1 of the polygon, walks backward along the second edge, and continues in this way for one circuit around the polygon, alternately walking forwards and backwards. If there are an odd number of edges, as in the 7-gon, then the walker ends up facing backwards on the first edge after the last internal turn. This indicates that the sum of the internal angles should be an odd number multiplied by 180°.

Note also, in the example shown, the turns through the internal angles are all counterclockwise, while the circuit around the polygon proceeds clockwise. We can see what the total sum should be if we imagine that the edges are instead all laid out along a straight line, as shown on the bottom of the figure. Then each of the seven turning angles will be half-circles or 180° and their sum will be 7·180°. However, if we bend the path back to the shape of the polygon by rotating the right end clockwise to coincide with the left end, then all the angles will again be internal to the polygon and we will have added one total turn in the opposite direction, which must have measure – 360°. So the total sum of the internal angles of the polygon should be (which it is!)

$$7 \cdot 180° + (-360°)$$

Similarly, the general formula for a non-self-intersecting polygon with n vertices and turning number k will be

$$n \cdot 180° - k \cdot 360°$$

which is usually written

$$(n - 2k) \cdot 180°$$

Interestingly the same reasoning implies that this formula also holds for self-intersecting polygons with n vertices and turning number k, for example the star polygons denoted {*n*/*k*}. This is the case even for "compound" star polygons where *n* and *k* share a common factor other than 1 and the star polygon is therefore two or more separate continuous paths, or even for other more complicated polygons. However, we stress this is not a proof, but a heuristic explanation. A proof would entail showing that the angle sums are invariant for a given number of vertices and turning number, even for the "degenerate" case of the edges in a long line. On the other hand, if that invariance has been proved, then the long-line can be seen as an extreme case, which yet simplifies the understanding of many unusual polygonal shapes. Questions about such polygons can inspire productive additional investigations by students.

Some of the polygons in Figure 3 may cause difficulties for students. Figure 7(a) depicts a concave quadrilateral with the directions of the external angle walker, who establishes the turning number or sum of the external angles of the polygon in red, and the directions the internal angle walker faces while alternately walking forward and backward in blue. The directions faced by the "adding machine" students are shown to the side along with their transitions that match those of the corresponding walker. Figure 7(b) shows a crossed quadrilateral. Two of its "internal" angles appear to be on the outside. This is so their directions continue to be counterclockwise. Its turning number is 0 and the sum of its "internal" angles is therefore (4 – 2· (0))180° = 4·180°, or 4 half circles.

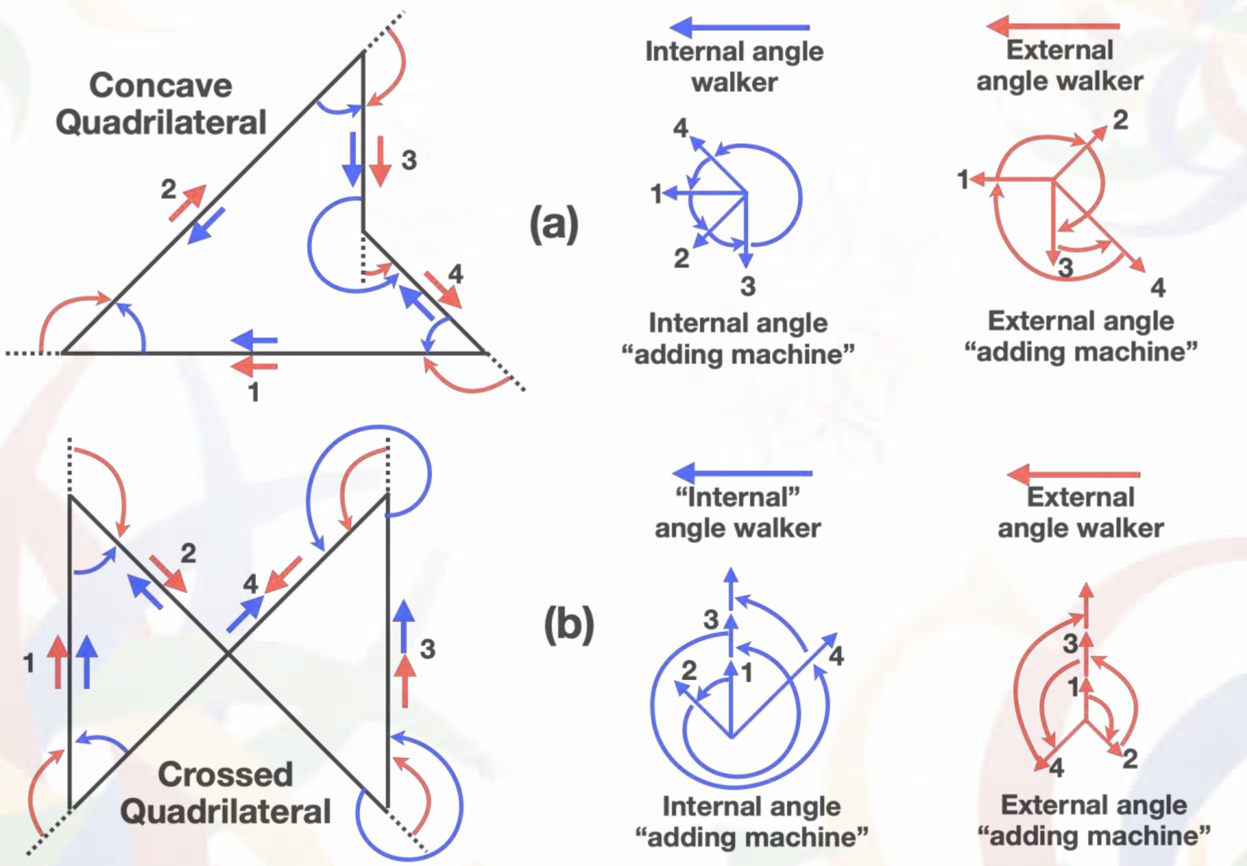


Figure 7. (a) *A concave quadrilateral. (b) A crossed quadrilateral.*

A detailed exploration of issues related to finding the sum of internal angles of more complex polygons can be found in [1] by De Villiers which connects the angle sum problem to use of the interactive computer software LOGO. Other references include [6] by Touval and Westreich which uses the alternating walking forward and backward model for summing the internal angles but limited to the cases of a triangle, pentagon, and five-pointed star (I was unaware of that reference until writing up this set of activities). Other references detailing educational activities are [5] which uses GeoGebra software to analyze the internal angle sum of star polygons, and Martin Gardner's letter [3] which proposes sliding a matchstick around drawings of various polygons and observing how it rotates as it turns through the internal angles. However, these activities do not connect angle sum with dance, movement arts, and planetary motion, or winding and turning number.

I began teaching this activity in the early 2000s in a Math for Elementary Education class for pre-service elementary and middle school teachers. I also used it to create a short dance version in 2008 and 2009 in two separate performances that was further developed and performed in 2015. When teaching it at a conference an Exploratorium science museum Staff Teacher, Lori Lambertson, suggested the placement of a student at the center of the polygon to physically act out the overall rotation sum, and we find that element to be a very helpful addition so include the student "adding machine" whenever running the workshop [2].

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[1] De Villiers, Michael. "From 'TO POLY' to generalized poly-figures and their classification: A learning experience." International Journal of Mathematical Education in Science & Technology. July 1989, Vol. 210, No. 4, pp 585-603. https://www.researchgate.net/publication/232950579_From_'TO_POLY'_to_generalized_poly-figures_and_their_classification_A_learning_experience

[2] Exploratorium. "Walking Polygons." Geometry Playground. 2010. http://web.archive.org/web/20101118084340/http://www.exploratorium.edu/geometryplayground/Activities/walkingpolygons.php

[3] Gardner, Martin. "Match This" in "Reader Reflections." Mathematics Teacher, Vol. 91, No. 4, pp 362-364.

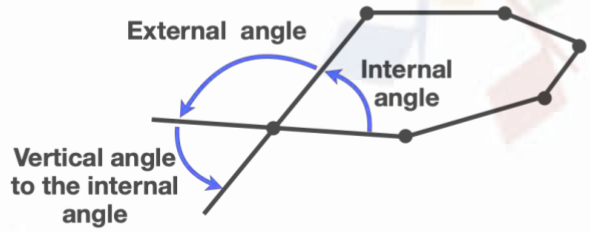
[4] Grünbaum, Branko. "Are Your Polyhedra the Same as My Polyhedra?." *Discrete and Computational Geometry*. Springer, 2003. https://faculty.washington.edu/moishe/branko/Your%20polyhedra,%20my%20polyhedra/grun.pdf.

[5] Peterson, Blake E. "A New Angle on Stars." Mathematics Teacher, Vol. 90, No. 8, Nov. 1997, pp 634-639.

[6] Touval, Ayana and Westreich, Galeet. "Teaching Sums of Angle Measures, a Kinesthetic Approach." Mathematics Teacher, Vol. 96, No. 4, April 2003, pp 230-233.

Polygon Vocabulary

Polygon – (sometimes called a *simple* polygon) a closed sequence of at least three line segments or edges in the plane, each sharing one vertex with the prior segment in the sequence and the other vertex with the following segment, and such that the edges intersect only at their endpoints.



Internal and external angles – For a polygon with a clear interior and exterior, the internal angles are formed by two adjacent edges in the interior, and the associated external angle is supplementary to it. An interior angle and its associated external angle have sum 180°. Although an angle measured in the clockwise direction is said to be negative and positive for one measured counterclockwise, in this activity and description we will generally avoid that distinction and consider the absolute value of angle measure, since many students at the 6^{th} to 8^{th} grade levels may not be familiar with negative numbers or angle measure.

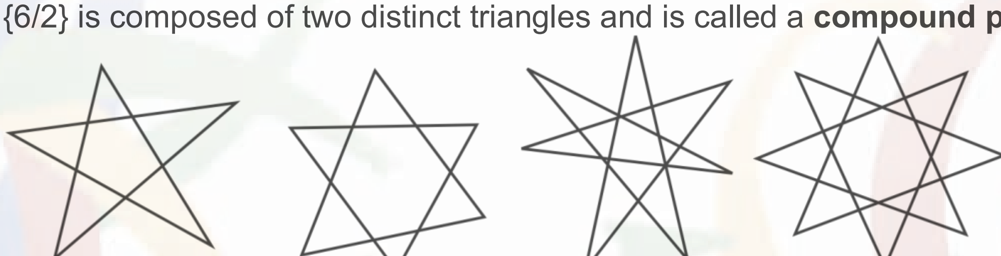
Convex polygon – a polygon with the property that any line containing an internal point and not containing an edge intersects the polygon sides in exactly two points.

Concave polygon – a polygon with the property that there exists a line that contains in internal point and intersects the polygon's edges in three or more points.

Self-intersecting or complex polygon – a closed chain of polygonal edges at least a pair of whose edges intersect in non-endpoints.

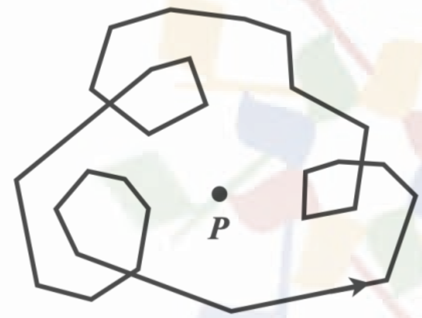
Crossed quadrilateral – a quadrilateral with a pair of edges intersecting at non-endpoints. Such a polygon may be tricky to decide on which are the internal and which the external edges. See the example in Figure 7.

Star polygon – a polygon with *p* vertices that are those of a convex *p*-gon and *q* edges, $q \geq 2$, each connecting the *k*th vertex in the sequence to the *k+q*th vertex. This polygon is denoted with the Schläfli symbol {*p*/*q*}. For example, the five-pointed star is denoted {5/2}. Note that {6/2} is composed of two distinct triangles and is called a **compound polygon**.



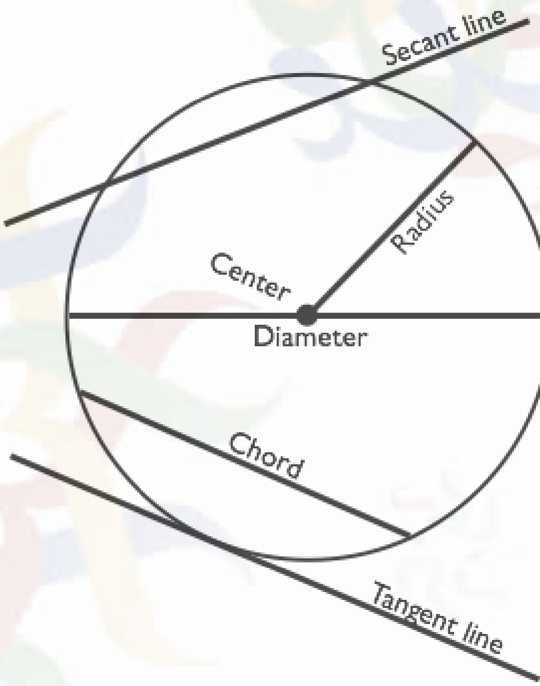
{5/2} {6/2} {7/3} {8/3}

Winding number – the number of times a closed path in the plane such as a closed polygonal sequence of edges circles counterclockwise around a specified point.



Turning number – (or rotation number) is the number of rotations the tangent to a smooth closed plane curve (or a polygon) executes in the counterclockwise direction around the curve. This is consistent with the standard definition of counterclockwise turns as positive. One can think of this as the number of times a person walking around the path in the counterclockwise direction, and always facing the direction of the path or the polygon's edges circles around their own center. This example has winding number 1 around the point *P* and turning number 4 since it makes one circle around *P* and three more smaller circular loops. In these activities we calculate the turning number as the total clockwise turning of the path to stay consistent with the direction in which students walk the internal angles. At a polygonal vertex, which has no tangent (or seems to have two!), we assume the tangent turns to the next edge in the direction that makes the turning angle's absolute value less than 180°.

Circle Vocabulary



The word “**circle**” comes from the Greek word *kentron*, which means peg or sharp point, and is probably related to the how one may draw a circle by placing a peg in the ground, attaching a rope to it, and moving the end of the rope around the peg.

radius: a line segment from the center of the circle to the circle. From the Latin word radius, meaning the spoke of a wheel.

diameter: a line segment through the center of a circle, with endpoints on the circle. From Greek *dia*, “across” or “through,” and *metron*, “a measure.”

Pi – the ratio of the circumference of the circle to its diameter. In decimal form, pi or π is equal to the irrational number 3.141592653589….

circumference: the distance around the boundary of the circle. From the Latin *circum*, meaning around, and *ferre*, meaning “to bring.”

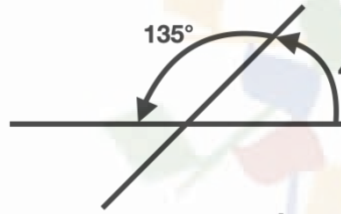
secant: a line intersecting a circle in two points. From the Latin *secare*, meaning “to cut.”

tangent: a line in the plane of a circle and intersecting a circle in one point. From the Latin *tangens*, “touching.”

chord: a line segment with both endpoints on the circle. From Greek *khorde*, “gut” or “string.”

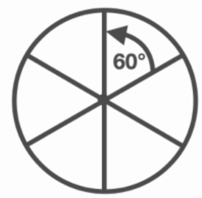
arc- a continuous section of a circle.

Angle Measure Review



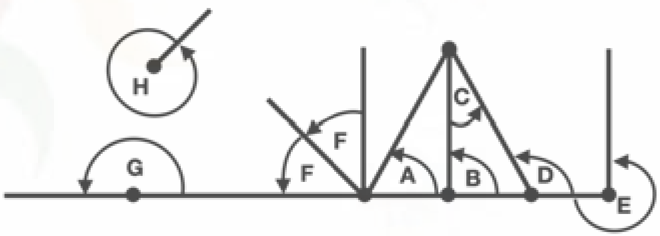
An angle is the opening between two intersecting lines. Angles are commonly measured in "*degrees*" using a base 60 system that was developed in ancient Mesopotamia thousands of years ago. In that system The "angle" measure of a straight line is 180 degrees, written 180°, and that of a circle is 360°. In the diagram to the right notice that 45° + 135° = 180°.

A "*right angle*" is like the corner where the wall meets the floor, and measures 90°, and half of that angle is 45°.



A circle divided by three lines into six identical areas produces angles of measurement 60°, as shown on the right.

1. Match the angle measurements with the closest appearing angles:



30°: ____ 45°: ____ 60°: ____ 90°: ____ 120°: ____ 180°: ____ 270°: ____ 360°:____

An *acute* angle is between 0° and 90°. An *obtuse* angle is between 90° and 180°.

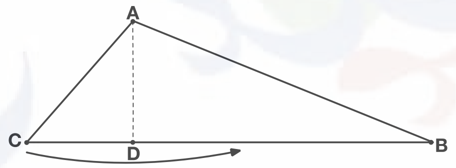
2. Draw a triangle with one right angle
and angles of 30° and 60°:

3. Draw a triangle with one right angle
and two angles of 45°:

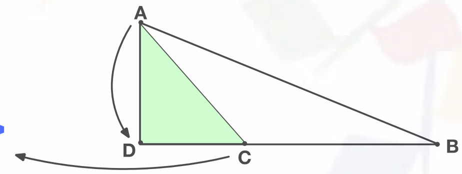
4. Draw a quadrilateral, a shape with four sides, that has
two right angles, one acute angle and one obtuse angle:

Summing the Internal Angles with Origami

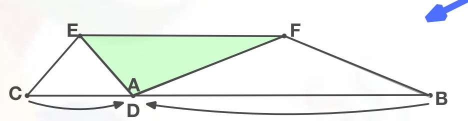
Cut out a large paper triangle. This demonstration can be accomplished with any such triangle and does not need fold lines drawn in. Let the longest edge of the triangle be side BC, with vertex A opposite it, as in the diagrams below.



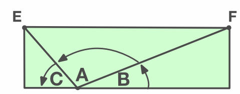
Valley fold point C to edge CB so that the fold includes point A. This pinches point D, the foot of the perpendicular through A, and creates the fold line AD.



Unfold point C and then valley fold point A down to point D.



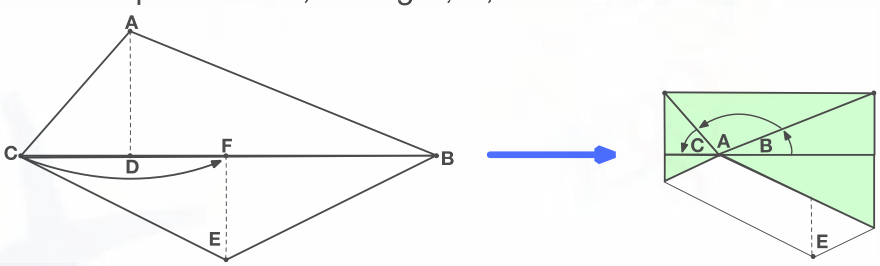
Folding A down to D creates fold line EF, where E and F are midpoints of AC and AB. Valley fold C to D and also valley fold B to F.



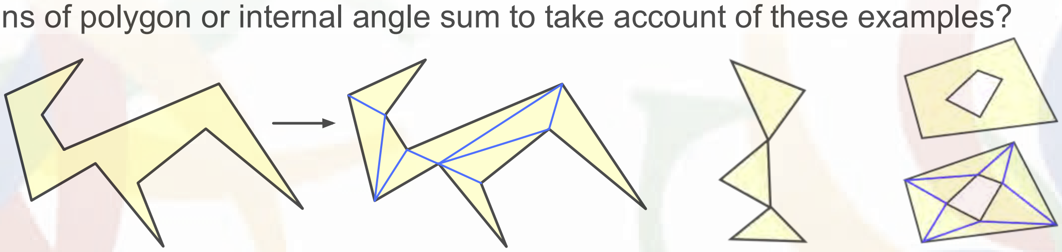
This now demonstrates that the sum of angles at points A, B, and C is 180° or a half circle.

A formal proof that the sum of the angles at points A, B, and C is 180° can be given using the observation that E and F must be the midpoints of sides AB and AC respectively, so that triangles DCE and DBF are isosceles.

For a quadrilateral, you may fold along the diagonal line CB and make a perpendicular fold line AD to that diagonal, then fold A, B, and C to D. Repeat these steps for the other half of the quadrilateral, folding E, C, and B to F:



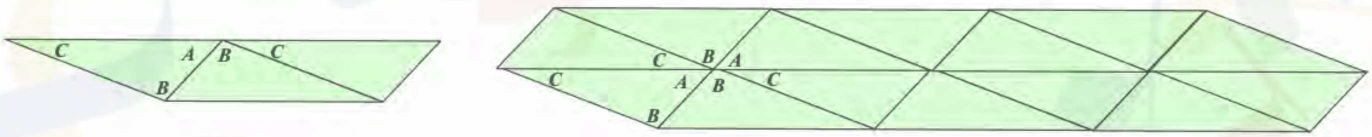
However, even complicated concave but non-intersecting polygons, (A) below, can be separated (or cut) into triangles which all have their vertices on the polygon boundary. Can you explain why such a polygon with n vertices will have internal angle sum $(n-2)180°$? The "polygon" (B) has $n = 7$ but internal angle sum $3 \cdot 180°$, and (C) with a "hole" in the center has $n = 8$ but internal angle sum $8 \cdot 180°$; how might we change the definitions of polygon or internal angle sum to take account of these examples?



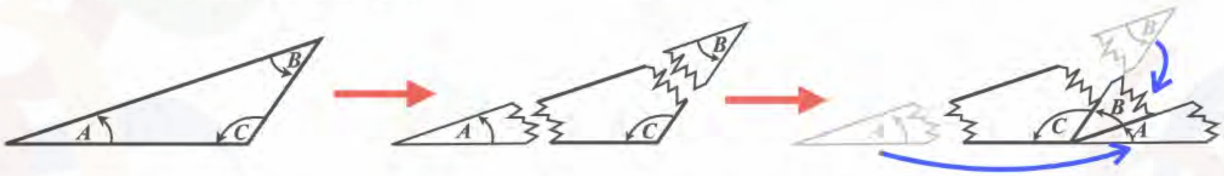
(A) (B) (C)

Summing the Internal Angles with Tessellations

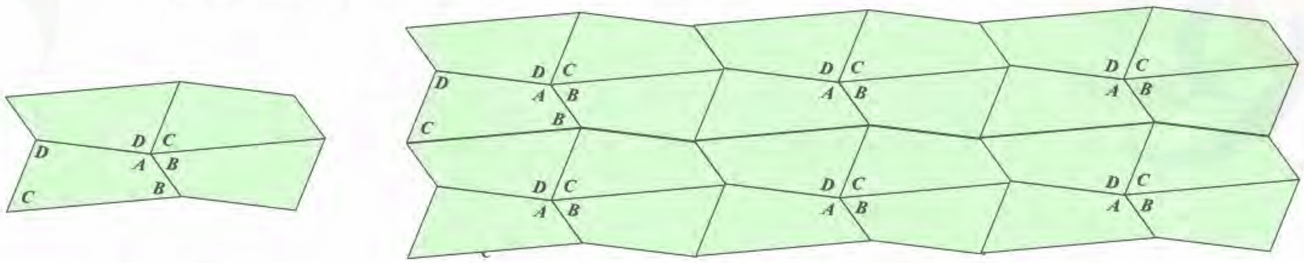
We can make three copies of any triangle and place them together so that the three angles at points A, B, and C are adjacent and the edges AC fall in a line. We can then use more copies to create as large a tessellation as we like:



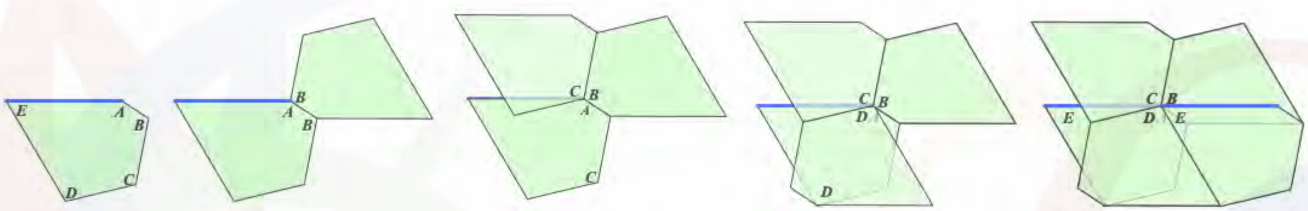
Another way to illustrate this is to tear two of the corners of the triangle off and assemble them next to the third angle:

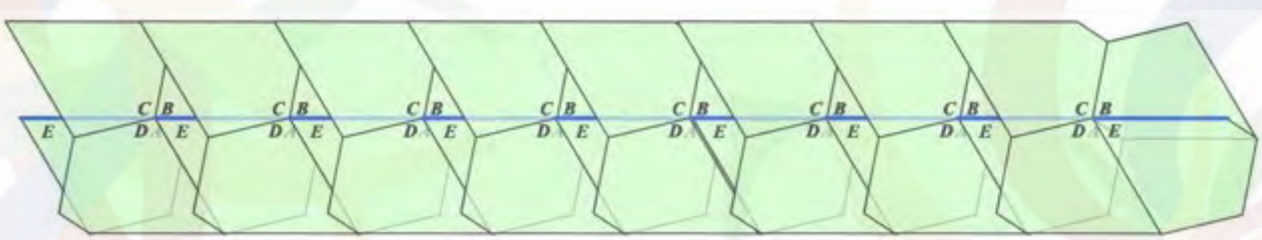


We can create a similar tessellation with multiple copies of any quadrilateral, demonstrating that the sum of the internal angles is 360°, or a full circle:



We can do something similar with multiple copies of a pentagon, though in this case the design created is an unusual “overlapping” tessellation. Here the blue line indicates the sum of the internal angles is $3 \cdot 180° = 540°$, or one and a half circles:





How is a Polygon Like a Circle Checklist

Individual Work

Work Process

- [ ] Student works actively on the assignment.
- [ ] Student works well with other dancers.
- [ ] Student takes part in presentation.

Dance Performance

- [ ] Student dances with clarity and aritculation.
- [ ] Student's energy level supports theme of work.

Group Work

Choreographic Structure

- [ ] Clear beginning, and coherent transition between it and next section.
- [ ] Clear ending, and coherent transition between it and prior section.
- [ ] Group solves choreographic problems that arise.
- [ ] Movement choices are inventive rather than imitative of classwork or other's work.
- [ ] Unified theme that the choreography elicits and supports.
- [ ] Movement choices lead from one to another efficiently, logically, and effectively.

Dance Performance

- [ ] Energy levels support the theme of the work.
- [ ] Movements are executed with clarity and articulation.
- [ ] Use of space (levels, stage areas, paths, etc.) supports the theme of the work.
- [ ] Composition uses rhythm clearly - or is creatively arrhythmic.
- [ ] Effort qualities (weight, time, focus, shaping) are varied and clearly executed.

Mathematical Exploration

- [ ] Angle summation properties are clearly and correctly examined and communicated.
- [ ] Angle and turning concepts are incorporated in movement tasks appropriately, creatively, and thoughtfully.
- [ ] Mathematical and artistic ideas are mutually supportive in final movement tasks.
- [ ] Final composition explores counting concepts, using them to support the theme of the composition.

Work Process

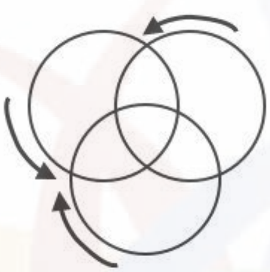
- [ ] Group works actively on the assignment.
- [ ] Group works well together.
- [ ] Group completes and presents composition
- [ ] After feedback, the group's work shows growth or improvement.

The Circle in Dance

There are many ways in which circles are used in dance. For example, dancers might follow circular floor patterns, turn their entire bodies in circles, or circle their limbs or other body parts. Dancers may also form circular shapes with their arms, legs, or entire bodies. Dance sequences might repeat in time, creating a circular phrasing, and the music used for dance might also repeat in a circular fashion. Props used in dance are also often swung or turned in circles, and costumes may be circular or employ circular motifs.

Floor pattern.
Dancers often follow circular paths. For example, the folk dances of many cultures are performed in a circular floor pattern. Below is the Austrian Round Dance, in a painting by Thomas Hans, from http://en.wikipedia.org/wiki/Austrian_folk_dance.

Try this: have your students work in small groups.
Teach them a simple circular folk dance.



Try this: have the students design circular floor patterns on paper, the translate them into simple walking or skipping sequences. For example, three dancers might skip around the three circles in this diagram, swinging around each other and switching circles, when they meet another dancer at one of the intersection points.

Body circles.
There are several ways to circle the entire body:
Circles around a vertical axis are turns or pirouettes.
Circles around left-to-right axis are somersaults or flips.
Circles around a front-to-back axis are cartwheels.

Pirouettes, somersaults, and cartwheels all can be tricky to do, and can lead to injuries. It is best to have someone with some dance or gymnastics training lead the students in these. But many students will already be accomplished at them, and they are exciting to do, to watch, and to learn.

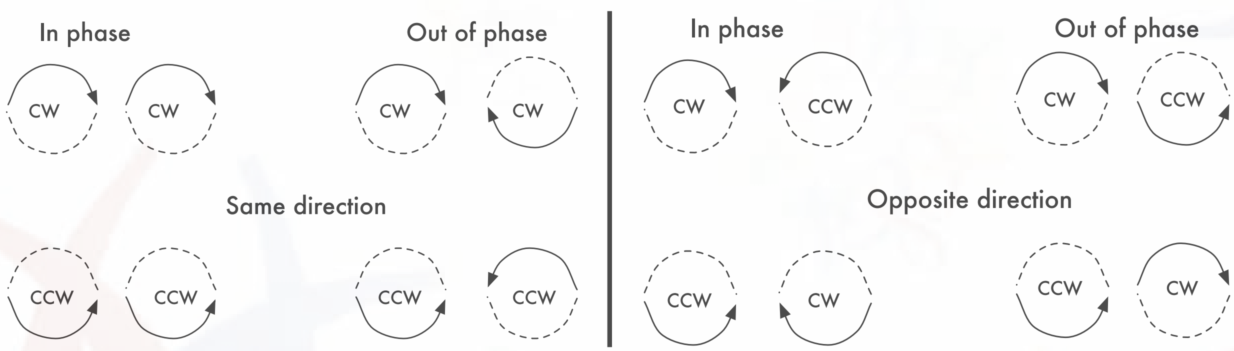
In addition, certain partner dance forms involve two dancers creating quick turns around a common axis. For example waltzes and polkas both involve a common axis, and salsa partnering often involves both duet turns and solo turns.

Windmills

Briefly, the activity here involves the following:
Have your class explore the many ways we can move our entire bodies or parts of our bodies in circles.

1. Circle the hands at the wrists, hold the hands near the shoulder and circle the elbows, circle the arms in large circles, and while sitting down circle the feet.
2. Find ways to make circular paths with other parts of the body.
3. Find ways to move in circular floor paths.
4. Explore properties of circles.

Background.
In this activity, students explore several ways of moving limbs and bodies in circular patterns, including swinging both arms in and out of phase with each other, then create sequences of group shapes that exhibit the symmetries. Students might also experiment with dancing with single sheets of 8-1/2 by 11 paper, which can be swung in circular motions that keep the hand pressed against the paper and thus keep it from falling to the floor. They will then construct and perform simple dance phrases using the circles and paper. The diagram below shows ways of windmilling the arms in and out of phase.



The joints of the body are such that it is possible to move the arms in circles in several ways.

Try this: Have the students work in groups. How many ways can they find to make circles in the air with different body parts. Create a sequence of circular movements, practice with your group, then have groups demonstrate for each the class.

Try this: The entire class can do this while seated. Hold both hands in front of you.

1. Circle both hands clockwise, in unison (both hands will be to the right at the same time, and to the left at the same time.)
2. Circle both hands counterclockwise, in unison.

3. Circle the left hand clockwise and the right hand counterclockwise, so the hands come together directly in front of you.
4. Circle the right hand clockwise and the left hand counterclockwise, so the hands come together directly in front of you.
5. Circle both hands clockwise, but half a cycle apart; that is, the hands will come together directly in front of you.
6. Circle both hands counterclockwise, but half a cycle apart; that is, the hands will come together directly in front of you.
7. Circle the left hand clockwise and the right hand counterclockwise, but half a cycle apart, so the hands never come together directly in front of you.
8. Circle the right hand clockwise and the left hand counterclockwise, but half a cycle apart, so the hands never come together directly in front of you.

9. Have each group invent a sequence of hand circles, practice them together, then demonstrate to the class.

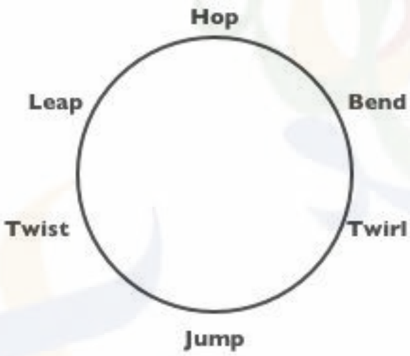
10. Make sure all students can stand and have enough space around them to be able to swing their arms freely without hitting other students. Try each of the activities 1-8, swinging the arms, not just the hands, in front of you. Each is possible.

11. While swinging the arms, the students may swing one arm behind and one in front, in a variety of ways. Have them try this as well.

12. Have the students make up a sequence of arm swings, make transitions happen fluidly, practice them as a group, then demonstrate to the class. Provide them musical accompaniment.

Circular shapes.
The arms and legs and torso can all be used to produce circular shapes.

A circular shape in a Bulgarian Women's dance.



Circular phrasing.
A circular movement phrase is simply one that repeats itself. We can create a circular diagram for such a phrase. To the left is a diagram for a phrase that includes several movements. Have the students pick a starting point and figure out how to repeat the phrase either in the clockwise or counterclockwise direction, performing the entire phrase at least three times. Have them make up and diagram their own movement phrases.

Prop Circles.
Many props are most naturally swung or spun in circles. For example, hula hoops are normally spun around the central axis of the torso but may be spun in other ways as well. Basketballs or other balls spin around their own axes or may be rolled up and down the arms or along the floor. Other common props are whipped in circles tied to the end of sticks.

Poi is the name of performance prop originating among the Maori people of New Zealand, in which a ball is held at the end of flexible material and swung in circles. Often the performer holds one poi in each hand. A simple and inexpensive – and safe – way to learn to twirl poi is to use a long sock with another sock stuffed down to the toes of the first. If the student twirls this and hits his or her head with it, it won't hurt! Fire dancers and jugglers often learn complex circular patterns with such props.

Try this: have the students bring extra pairs of socks, and practice poi moves.

Costumes. Below, a woman swirls a costume from Jalisco in a large circle around herself. (From http://en.wikipedia.org/wiki/Baile_Folklorico)

Turns in ballet and other dance forms

Ballet terminology is traditionally composed of French words due the formalization of the dance form in France. However, dancers and choreographers often use words like turn, rotation, and revolution to refer to the turning movements which are an essential part of most dance forms. Rotation usually means turning around one's own center and revolving usually refers to locomotion around a stable center. Duet turns are common in many folk and social dance forms and usually involve partners turning together around a common center. Examples are waltz, polka and hambo. Hambo is unusual in that partners dance the same steps but one beat apart in a 3 beat cycle. Many social dance forms such as tango or salsa involve partners turning both together and separately.

Folk dance forms often involve dancers moving in a large circle and turning (revolving!) around the circle's common center. Sometimes the dancers additionally turn around their own center or their and a partner's common center.

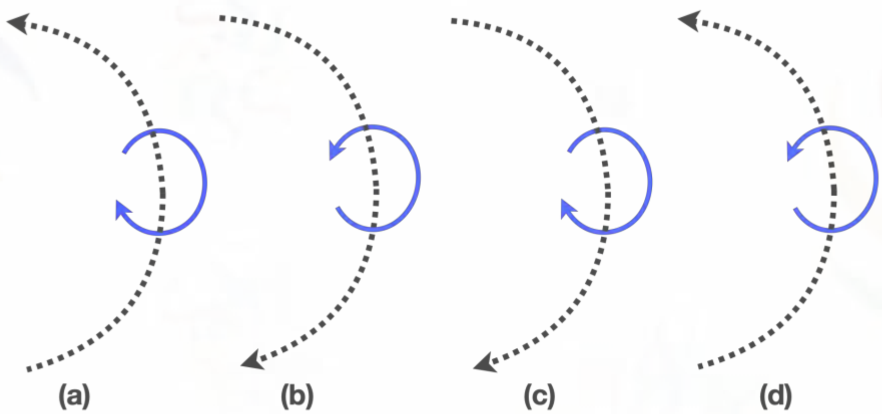


Figure 8. *Four types of dance turn floor patterns. (a) "Natural waltz" pattern.(b) "Reverse waltz."(c) and (d) Sometimes used in tango or more free form dance work.*

Figure 8 shows several types of partner turn patterns for duets moving around a common center at the same time they circle the dance floor. Figure 8(a) and (b) are like retrograde motion by astronomical bodies, and Figure 8(c) and (d) are called prograde motion in astronomy. But Figure 8(a) and (b) are also common waltz patterns known respectively as natural waltz and reverse waltz patterns. Figure 8(c) and (d) may be used in tango or more free form dance work where they are called "moving backwards."

Virtuosic individual turns are performed in large numbers of dance forms from classical Indian dance to Sufi whirling to break dancing. Two types of common turns in ballet are known as en dehors and en dedans:

En dehors – turn to the right on the left leg with the right leg lifted usually in passe position (with the foot at the knee). The left leg is called in ballet the "standing leg," and the lifted right leg is called in ballet the "working leg" or "moving leg." This would be an "outside turn" in most ballet teachers' use of the term.

En dedans – opposite of en dehors. So turning to the right on the right leg with the left leg lifted in passe, an "inside turn."

Note there are thus four possibilities for standard ballet pirouettes: inside or en dedans to right or left, outside or en dehors to right or left.

These turns take careful training to master but can be beautiful to watch. Consider having students or dancers demonstrate or teach simple (and safe!) turning patterns. Find videos of dance forms with virtuosic turns to show to students.

Rotation and Orbits in Astronomy

Planetary bodies often exhibit several types of motion called retrograde and prograde that are related to the work students do in this set of activities. Retrograde and prograde motion tend to have two usages in astronomy. If one were to look down at the solar system from above the sun's North pole – in the same approximate direction as the Earth's North pole – then the sun would appear to rotate counterclockwise around its North/South axis, and the planets would appear to rotate counterclockwise around the sun. The planets are thus said to rotate in prograde fashion around the sun since they orbit in the same direction as the sun turns. This is the pattern shown in the student handout on the next page.

But the earth also rotates around its own axis counterclockwise when viewed from the solar system's (or Earth's) North pole, so that rotation is also known as prograde with respect to Earth's orbit around the sun. Triton, Neptune's largest moon, rotates in retrograde around Neptune, that is clockwise. Triton is the only moon in the solar system that rotates in retrograde with respect to its planet. Bennu, the large asteroid that a NASA probe returned rock samples from in 2023, rotates in retrograde with respect to its counterclockwise rotation around the sun.

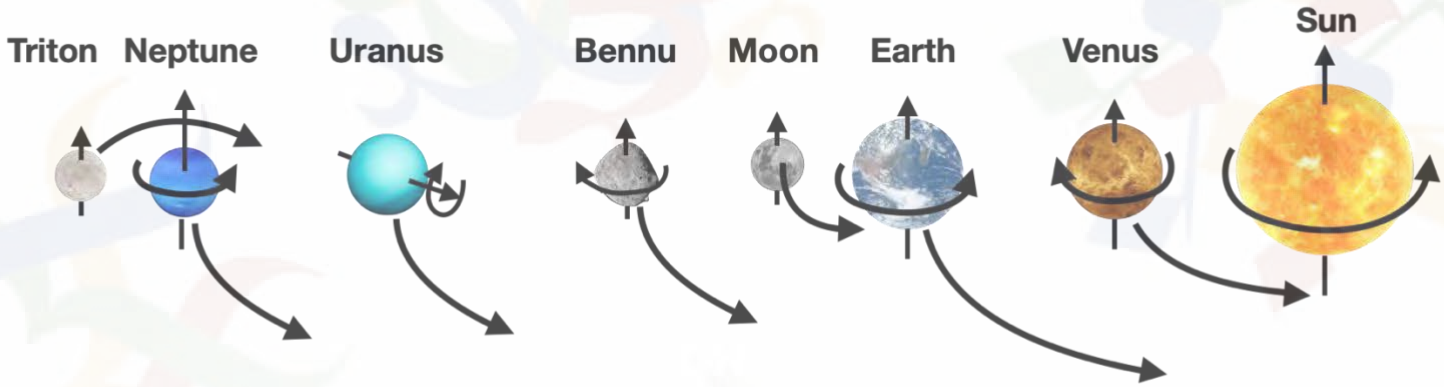
Of the eight planets in the solar system, only Venus and Uranus are not rotating in prograde around their own axes. Venus rotates in retrograde, and Uranus rotates with its axis of rotation almost parallel to the plane containing the orbits of all the planets. This would be like a dancer rolling on the floor while at the same time revolving around a central dancer in the middle of the stage! But the Uranus axis of rotation itself always faces the same direction, so one pole faces the sun for half its 84 year orbit around the sun, and the other pole faces the sun for the other half. Additionally Uranus rotates around that axis once every 17 hours, the length of its "day."

The Earth's moon appears to have no rotation on its axis since one side faces the Earth constantly. However in the 27.3 days it takes for the Moon to orbit the Earth the moon will have turned once on its own axis, a phenomenon known as synchronous orbit. Triton's retrograde orbit around Neptune is also synchronous.

Note that the activity of walking the internal polygon angles is like retrograde planetary motion, since the walkers rotated clockwise around a central point, while turning through internal angles in a counterclockwise direction. And the walking of the external angles is prograde since both are done in clockwise fashion in this activity. Students may be intrigued about the connections between the circling movements they executed in the activity and the rotations of heavenly bodies. The next page is a handout for an additional activity that explores these connections. However, deeper connections with the ideas of the orbits of heavenly bodies can be explored when we pay attention to the force that controls those orbits, namely gravity, and the ubiquitous nature of our everyday experience of gravity as denizens of such a planetary body!

Names of group members:____

Spinning Like a Planet!



The sun, planets, moons, and asteroid shown are NOT to scale!

In your work walking around polygons you circled the polygons clockwise, turned through external angles clockwise, and turned through internal angles counterclockwise. Turning around your own axis in the same direction as how you circled around the center of a polygon, as you did for external angles, is known as **prograde** motion. Turning in the opposite direction, as you did for internal angles, is called **retrograde** motion. In the diagram above of some of the planets, moons, and the asteroid Bennu the arrows pointing upward are known as the North poles of the body. Note that all the planets move counterclockwise around the sun from the point of view of an observer above the North pole of the Sun and the solar system. The Sun also turns counterclockwise around its own axis.

1. Which are moving in retrograde to their orbit around the sun?

2. Which are moving in prograde to their orbit around the sun?

3. Uranus is unique among the planets in that its “North” pole is approximately horizontal with respect to its and the other planets’ orbits. What kind of human motion around a central Sun like point would imitate Uranus’s rotational motion?

4. Neptune’s moon Triton and our Moon are unusual in that one side of each always faces its planet, so it rotates on its own axis once for each orbit around the planet. Model this with a partner, one of you acting as the planet and the other as the moon. What was the experience like?

5. Act out planetary motion like heavenly bodies in our solar system. What did you learn or experience?

6. Act out the motions of astronomical bodies in an imaginary solar system and record your work and experience in diagrams and words.

7. Planetary orbits are determined by the force of gravity. But gravity is also the force affecting us most constantly on the surface of the Earth. Problems 5 and 6 ask you to illustrate the motion of heavenly bodies with your own movements. But can you create movements that help explore and illuminate how planetary motion is intimately connected to our everyday experience of how gravity affects us? For example, can you raise and then drop your arm so that it begins to move in a circle?

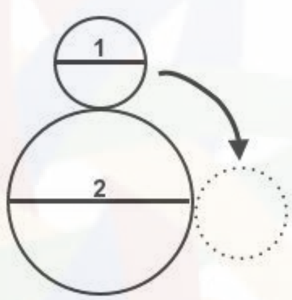
Puzzling Circles

In our polygon activities students have investigated what happens when they move around polygonal paths and return to their starting point, in the process accomplishing additional rotations. Here are some questions about circles that relate to what happens when they roll and rotate. They may seem simple but still require some thought. See if you can answer the following questions without trying them, then try them with wheels, quarters, or other circular objects.

1. Which travels faster with respect to the ground, the top of a rolling wheel or the bottom, or are they the same?

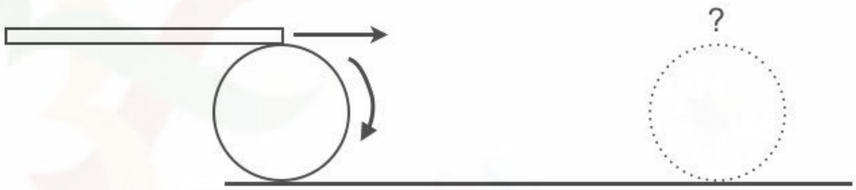
2. (Don't physically try this one!) Automobile speedometers calculate the car's speed based on the rate at which the axel is rotating. If you replace your car's tires with larger diameter tires, will the speedometer show a speed that is
(A) Less than the actual speed (B) The actual speed
(C) Greater than the actual speed (D) It depends on how fast the car is going

3. If we roll one quarter all the way around another quarter (the bottom quarter in the diagram stays fixed), then how many times will the image of Jefferson turn in a circle?



4. If we roll a circle or wheel with diameter of 1 around a larger circle of diameter 2, how many complete turns will the smaller circle have executed when it returns to its starting point?

5. Here is another rolling coin puzzle. Suppose a quarter (the lighter quarter) rolls around two stationary quarters (the darker ones). How many times will the rolling quarter rotate before returning to its starting point?



5. Suppose a long beam is rolled along the top of a moving wheel of diameter 1. If the wheel completes exactly one revolution along the ground, then how far will the beam travel in relation to the ground?

(A) 1 foot (B) 2 feet

(C) π, or approximately 3.1 feet (D) 2π, or approximately 6.3 feet

6. Make up some circling puzzles of your own!

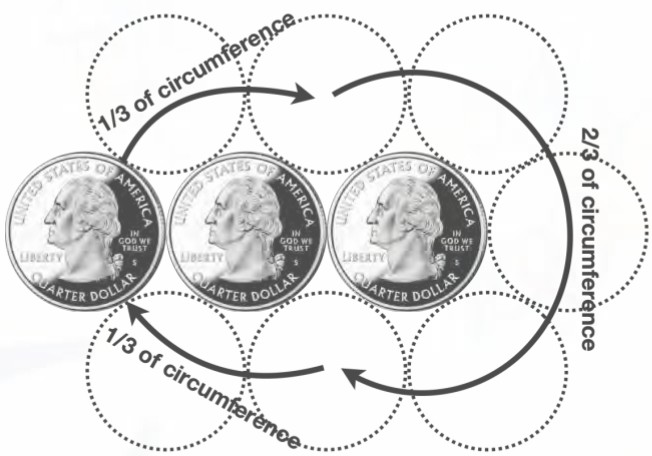
Puzzling Circles: Solutions

1. The rolling wheel. Believe, it or not, the top of a rolling bicycle wheel is traveling twice as fast, in relation to the ground, as the bike itself. And the bottom of the wheel touching the ground is momentarily stationary! This is because the relevant velocities are "additive." If the bike is traveling 10 miles per hour, then the wheel must be spinning at that rate, and the bottom of the wheel is traveling in the direction of its tangent, backwards, at 10 miles per hour. The two velocities cancel, leaving the point in contact with the ground with velocity 10 + (–10) = 0 mph. The top of the wheel is moving, momentarily, in the direction of its tangent, forward, so its total velocity is 10 + 10 = 20 mph.

For a detailed analysis involving calculus, with video clips, see. http://www.animations.physics.unsw.edu.au/jw/rolling.htm, produced by the physics department of the University of New South Wales.

2. For each revolution of the wheel, the speedometer will "think" the car has traveled less far than it has, so the speedometer will show a speed that is less than the actual speed.

3. If the coin were rolling along a horizontal line a distance equal to the quarter's circumference, then the coin would rotate one complete turn. Since the circumference is "bent" around one complete turn, the rolling coin will itself turn twice!



4. 2-1/3 times. The rolling quarter rolls along 1/3 of the circumference of the first quarter, then 2/3 the circumference of the second quarter, then 1/3 of the circumference of the first quarter again. So 4/3 of the circumference plus one additional rotation since its "turning number" is one, for a total of 2-1/3 times.

5. 2π, or approximately 6.3 feet. If the circle were stationary while it turned on an axle, the beam would move a distance equal to the circumference. But the circle travels to the right also a distance equal to its circumference, so the beam travels twice the circumference to the right.