MOMATH

NATIONAL MUSEUM OF MATHEMATICS

THE 2022 ROSENTHAL PRIZE
for Innovation and Inspiration in Math Teaching

Tranformation Sumo, A Geometry Game

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Transformation Sumo: A Geometry Game

Introduction

Transformation Sumo is a competitive geometry game for two players. During play, students will internalize vocabulary and concepts about geometric transformations. Afterwards, the game becomes a set of mathematical manipulatives to help students engage in follow-up activities that teach about transformation rules, compositions, and the concept of a group.

Part 1: Explaining the Game's Rules

Materials Required

For every pair of students:

- A game board and the two pages consisting of a total of 32 regular cards and 4 smaller focus cards (provided in the appendix of this lesson plan), ideally printed on card stock (regular paper will work, but not as well)
- Two pairs of scissors (to cut the cards out if this hasn't been done already)
- A large paperclip or medium-sized binder clip (to organize the cards after the activity is finished)
- Chips or coins (1 for the game, 3 for the extensions). Ideally, use chips with a different color on the top and the bottom
- One 6-sided die

Preparation and distribution will take 5-10 minutes of class time, depending on whether or not the cards are already cut out.

Have students cut out the cards, if needed. This will take about 8-10 minutes, give or take. It will be helpful to distribute the other materials during this time. If you teach the lesson to multiple classes, this only needs to be done once.

Setup

Tell your students to remove the "Focus cards" for now (we'll discuss their purpose later). Have students shuffle the cards and give four cards to each player.

All distributed cards are **public**, that is, they should be face up and visible to both players at all times. The deck should remain face down.

The setup should look something like this, although the board and the cards can appear anywhere on the table: [1]

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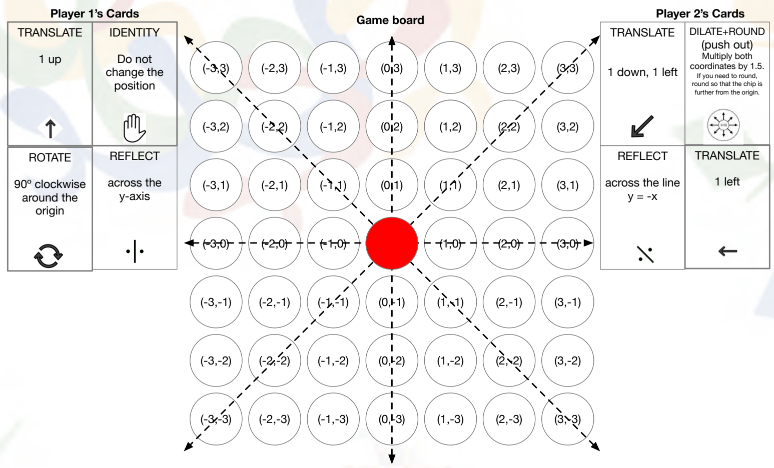
[1] There are a few very intentional design decisions regarding the game board. Firstly, points are discrete, large circles. This makes it easier to move the chips around. Secondly, the coordinates are written on each point. This



Turns

Players alternate turns. One player must go first; you can let this be decided by the rule "let the player who looked at their reflection most recently go first". Each turn involves the following steps:

- Roll the die. If it lands on 1-3, the player **must** play one non-bonus card. If it lands on 4-6, the player **must** play two non-bonus cards. After each card is played, another is taken from the deck, so one's hand should always contain 4 cards. Played cards go in a discard pile, face up. Then the other player takes a turn.
- Bonus cards can be played at any time, including during the other player's turn, and do not cost a move. The moment one is played, a new card is taken.
- A player cannot have more than 2 bonus cards in their hand. If the player receives a third bonus card, they must discard it and take another card.
- If the deck runs out, shuffle the discard pile. Then that becomes the new deck.

Victory

A player wins if, after their move, the chip is pushed out of the game board.[2]

Focus Cards

Focus cards are an optional mechanic intended for after the students have worked on the follow-up activities (see Part 4). They are not recommended for first-time players.

You place all four focus cards, face up, on the table. A player can take a focus card of their choice if they state aloud three or more cards, not including the IDENTITY or ORIGIN cards,

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makes finding patterns, and making connections between coordinate-based rules and transformations, far easier than on a traditional grid. Lastly, this format makes it easier to discuss lines. For instance, the line "y=1" can be expressed as "the line made up of points whose y-value is 1", and students can easily see all the points whose y-value is 1 on the game board.

[2] Hence the analogy with Sumo. In Sumo, each wrestler endeavors to push the other wrestler to the outside of the ring.

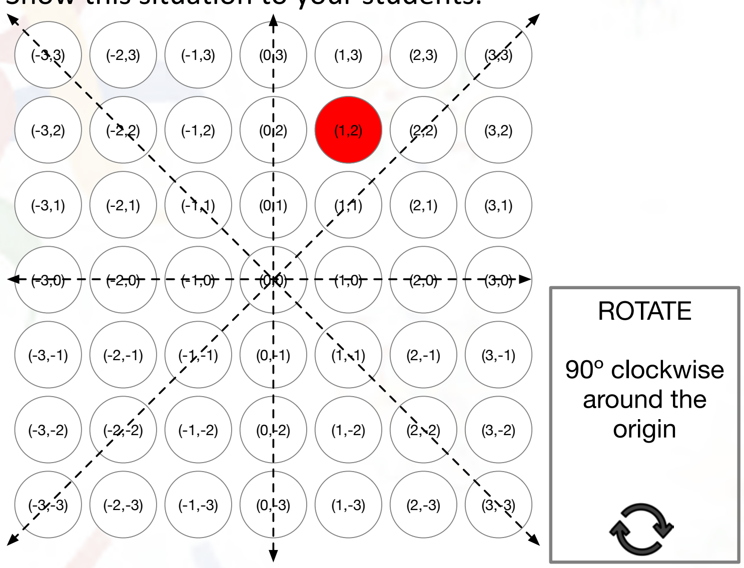
across both players' hands, whose combined effect results in the chip not being moved at all.[3] Focus cards cannot be taken if the chip is at the origin. Focus cards give significant powerups to the players who earn them, and incentivize students to pay attention to both players' hands.[4]

Part 2: Introducing Some Situations to your Students

Students will understand most of the game's cards fairly quickly, but there are some situations you might want to talk about. Which ones will be most helpful might depend on the prior knowledge of your students. All the pictures in this section are found as full-screen slides in the appendix.

Situation 1: Rotations

Show this situation to your students.



Ask your students: if this is the starting position and this rotation card is played, where does the chip move to? Try to get a sense of students' reasoning. You may want to coax out and emphasize certain ideas:

1. That an angle of 90º must be formed if we extend lines from the origin to the positions of the *original location* and its *mapping* (see the visualization below).

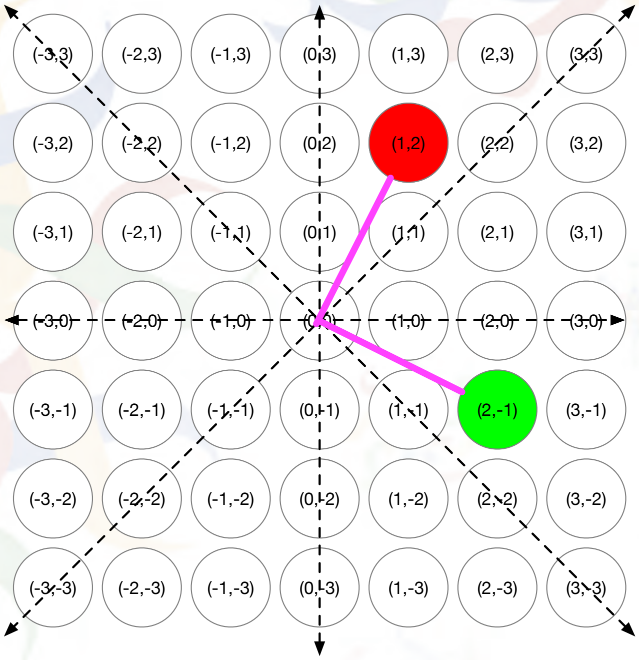
2. That rotation by 90º in either direction involves switching the x- and y- coordinates, and negating one of them, depending on whether the rotation is clockwise or

[3] A motivation behind this mechanic is to have students apply ideas about the composition of multiple transformations that they learn in Follow-Up Activities #2 and #3. An example of this mechanic is that a reflection across $y = x$, followed by a reflection across $y = -x$, followed by a rotation of 180º around the origin, results in the identity transformation, and the chip ultimately staying in the same place it began. In notation, $R_{180^\circ} \circ r_{y=-x} \circ r_{y=x} = I$.

[4] The metaphor of "Focus" both alludes to the underlying mathematical idea (simplification from complexity) and the powerup (a focused mind confers benefits).

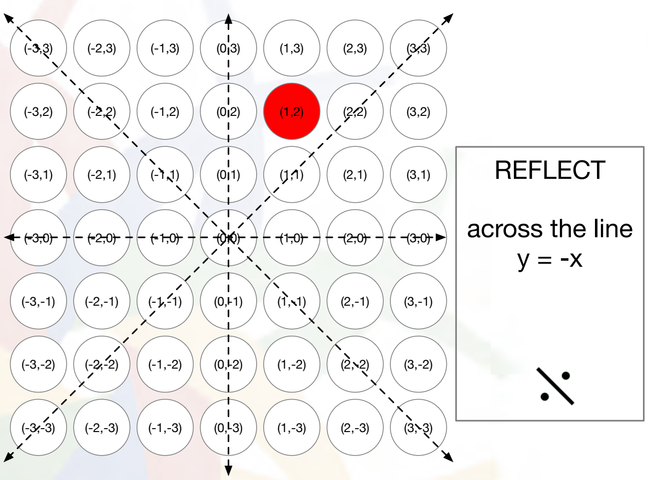
counterclockwise. In the case of clockwise rotation, the new y-value is negated; (1,2) becomes (2,-1).

The correct 90º clockwise rotation is shown below:



Situation 2: Reflection

Students will generally be proficient with reflecting across the x- and y- axes immediately, but they might not instantly have a good intuition about reflecting across other lines, such as $y = x$ or $y = -x$. It might be worthwhile to show the following slide (available in the appendix) and ask: where does the chip go if reflected across the line $y = -x$?

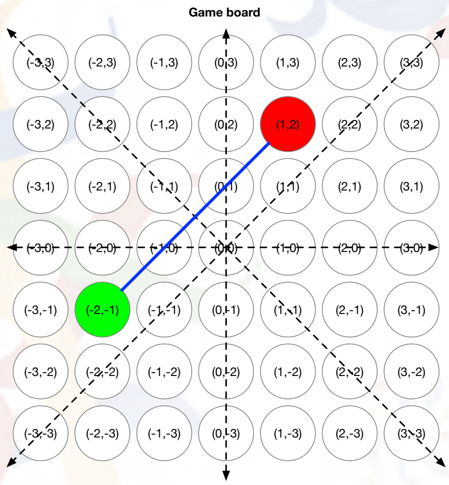


Again, you may want to emphasize certain ideas:

- The original coordinate and its mapping must be on a line that is perpendicular to the line of reflection.

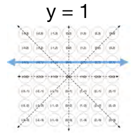
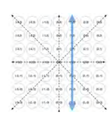
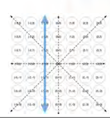
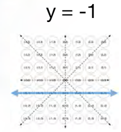
- The original coordinate and its mapping must be the same distance from the line of reflection.

The correct result of the reflection is shown below[5]:



Situation #3: Reflections of the form y=n or x=n

These cards may be worth discussing:

REFLECT across the line y = 1 	REFLECT across the line x = 1 	REFLECT across the line x = -1 	REFLECT across the line y = -1 

Make sure your students know where these lines are. You can state, "the line $y = 1$ is the line made up of all the points where the y-value is 1", and point to the line on the board. You may also want to draw their attention to the image on the bottom of the card, which illustrates its location.

Situation #4: Is victory possible?

Show the following situation to the class and ask, "can a player with these cards win if they play two cards during their turn?"

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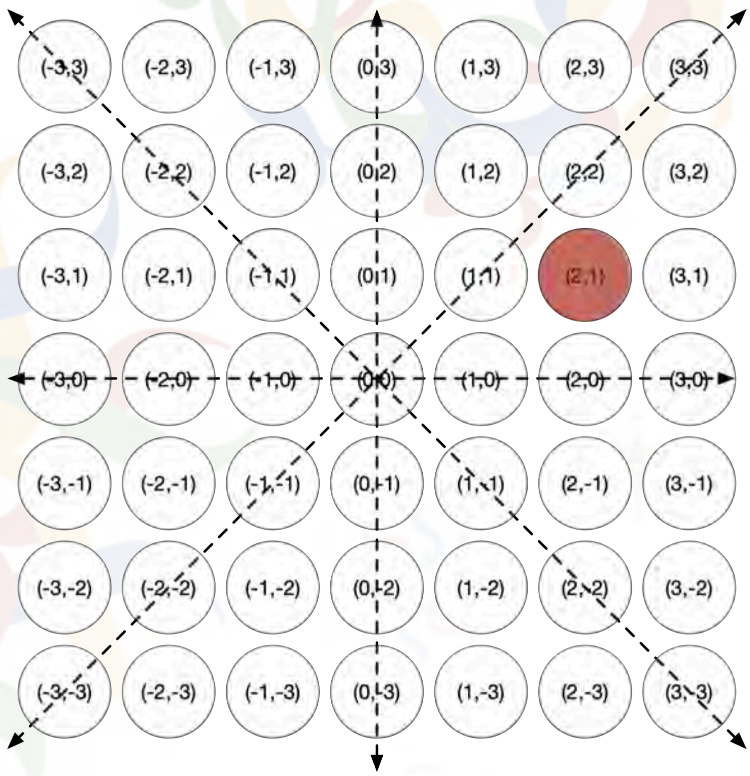
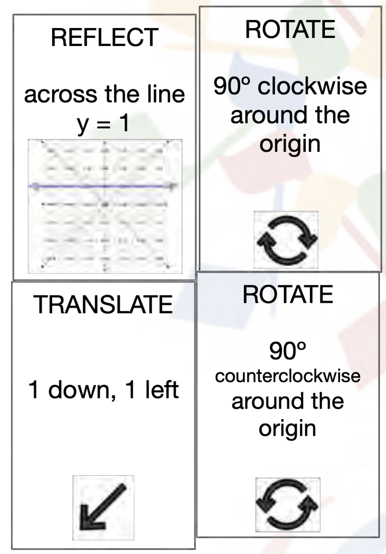
[5] Cards have symbols on the bottom (often from OpenMoji, an open-source emoji compendium, and sometimes original) that illustrate the transformations. The picture on this card shows the line y=-x, and a point and its reflection; it isn't a rotated division sign. The symbols are meant to assist recall and help English language learners manage the complexities of the game.





Can you win in two moves?

The answer is “yes”; if the player first rotates 90º clockwise, the chip moves to (1,-2), and if the player then reflects it across y=1, the point moves to (1,4), which is off the board.

Part 3: Helping students play the game

Tell students to begin playing. As you circulate, see what questions they have and what issues they are facing. Sometimes their questions will be answered by re-explaining concepts from the previous section, sometimes by reminding them about and clarifying rules, and sometimes by suggesting strategies.

Some questions you could ask:

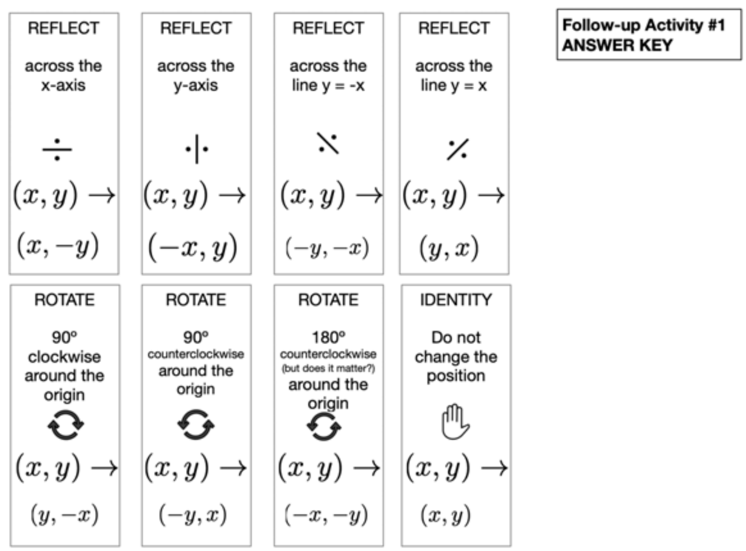
1. “Is there a way for you to win this turn?”
2. “Look at your opponent’s cards. Are there any cards you shouldn’t play?
3. “How do you rotate a point 90º again?” or “How do you reflect across the line y=x again?”
4. “Where is the line ______? How do you know?”
5. “Can you show me ___________?”
6. Look out for situations where a student doesn’t play a relatively complex transformation (such as rotation by 90º or reflection across a line like $x = -1$) and make sure they’re not avoiding it because they don’t know how it functions.

Part 4: Follow-up activities

Once students understand the game, its materials then become manipulatives that help students understand transformations more deeply. I think it's best to do these activities the day after the game is introduced and played. Each activity involves handouts that can be found in the appendix. With less comfortable students, you can structure a class period around Activities #1 and #2, and with more proficient students, you can go straight to Activity #3. After students internalize the concepts from Follow-up Activity #2 and #3, you can reintroduce the game, this time with the focus cards included.

Follow-up Activity #1: What's the rule for each card? (5-15 minutes)

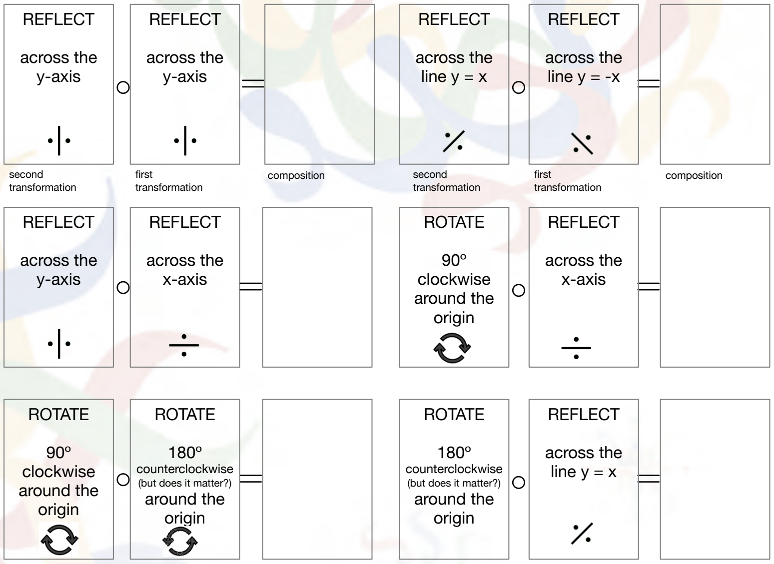
The idea of this activity is to write how to transform the coordinates of the original point into the coordinates of the transformed point. For instance, the rule for reflecting across the x-axis is to negate the y-value: $(x, y) \rightarrow (x, -y)$. The answer key is below. You should give 2 chips per board, and have students compare the coordinates of the original point and the transformed point by placing the chips on the two points.



Follow-up Activity #2: Composing transformations (10-20 minutes)

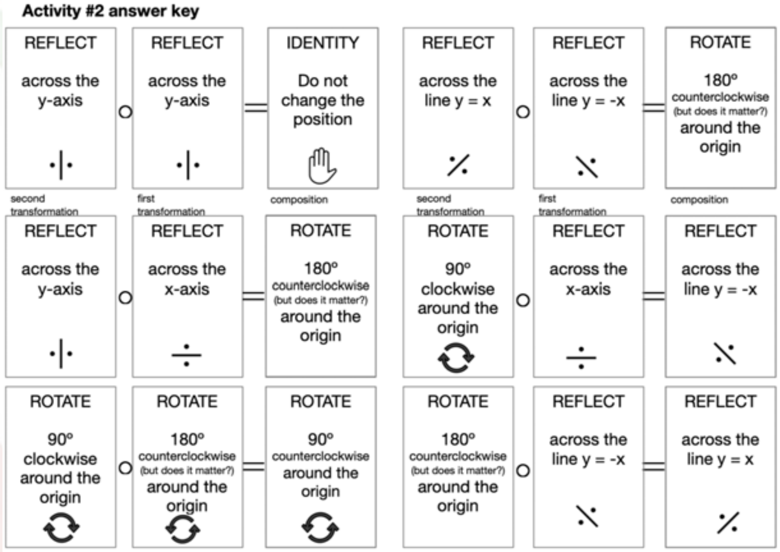
The idea of this activity is to have students figure out when a transformation is equivalent to the composition of two transformations.

Give each pair of students **3 chips**. They need to figure out what transformation is the result of playing one transformation card, and then a second transformation card. They will more easily see the composite transformation by putting chips on an original position, an intermediate position after one transformation, and a final position after both transformations. Emphasize the right-to-left order of the notation, which is used for the composition of functions throughout mathematics.



The answers are:

Activity #2 answer key



[6]

It is also worth writing the following question on the board as they do this activity.

Does the order of the transformation make a difference?
Always? Sometimes? Never? Explain.[7]

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[6] The task is at first relatively simple but the mixtures of reflections and rotations may be difficult for students. Teach them to slow down by keeping both the original position of the chip on the board as well as the position after both transformations, and then asking, "what single transformation does the work of both of these transformations?"

[7] Essentially, the answer is that the order of transformations does not matter if both are reflections or if both are rotations. But if one transformation is a reflection, and one is a rotation of 90º, then the order does matter (in other words, these transformations are not commutative). In more formal terminology, this means the underlying group of transformations is not "Abelian". This type of thinking is expanded in Follow-Up Activity #3.

Follow-up Activity #3 (40+ minutes): A “times table” for transformations.
Print out copies of the following “times table” for your students (a full-sized version is in the appendix). Put students in groups of 4 and give out two grids per group, but tell them that their goal is to collaborate to fill in the entire grid. In addition, you should provide 3 chips for each board.

First

Second

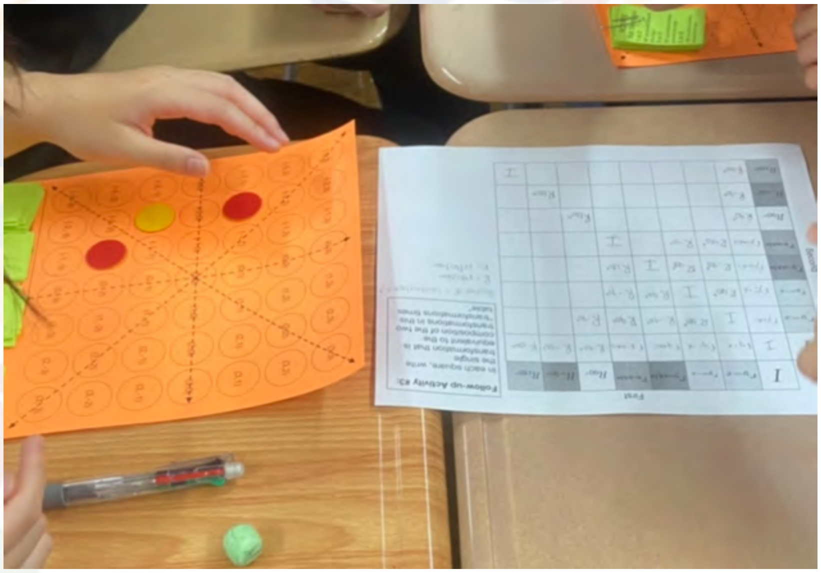
	I	$r_{y=x}$	$r_{y=-x}$	$r_{\text{y-axis}}$	$r_{\text{x-axis}}$	R_{90°	R_{-90°	R_{180°
I								
$r_{y=x}$								
$r_{y=-x}$								
$r_{\text{y-axis}}$								
$r_{\text{x-axis}}$								
R_{90°								
R_{-90°								
R_{180°								

Explain the notation. Capital R refers to rotation, and a positive angle measure refers to a counterclockwise rotation around the origin by default; -90º is clockwise rotation. Lowercase “r” refers to reflection, and “I” refers to the identity transformation.

In each square, put the transformation that is equivalent to the two composed transformations, first the one in the top row, and then the one in the left-most column[8]. Like in the previous activity, students can use 3 chips to visualize the beginning, intermediate, and final positions of a point. Because this task can be difficult, I recommend marking the original position with a

[8] The reason why this works is that we are taking a subset of the cards in the game that form what is known as a “group”; a set of objects that obey certain algebraic axioms. Specifically, this group is the Dihedral Group D4, also known as the “symmetry group of a square”. If a group is finite, it is possible to produce a “times table” like this, which is called a Cayley table, named after British mathematician Arthur Cayley (1821-1895).

chip, the second position by a different color/type of chip, and the third position by a chip that matches the first one. That way students can see the composite transformation more easily. This technique is shown in the photo below:



An answer key is below. You might wish to go through a few of the problems with your students. It also might help to mention (after they make some progress) that each row and each column of the table must contain each transformation only once, which will help students check their status. It is also important to tell students that they should distribute the workload, check each other's answers, and strategize together.

First

Second

	I	$r_{y=x}$	$r_{y=-x}$	$r_{\text{y-axis}}$	$r_{\text{x-axis}}$	R_{90°	R_{-90°	R_{180°
I	I	$r_{y=x}$	$r_{y=-x}$	$r_{\text{y-axis}}$	$r_{\text{x-axis}}$	R_{90°	R_{-90°	R_{180°
$r_{y=x}$	$r_{y=x}$	I	R_{180°	R_{-90°	R_{90°	$r_{\text{x-axis}}$	$r_{\text{y-axis}}$	$r_{y=-x}$
$r_{y=-x}$	$r_{y=-x}$	R_{180°	I	R_{90°	R_{-90°	$r_{\text{y-axis}}$	$r_{\text{x-axis}}$	$r_{y=x}$
$r_{\text{y-axis}}$	$r_{\text{y-axis}}$	R_{90°	R_{-90°	I	R_{180°	$r_{y=x}$	$r_{y=-x}$	$r_{\text{x-axis}}$
$r_{\text{x-axis}}$	$r_{\text{x-axis}}$	R_{-90°	R_{90°	R_{180°	I	$r_{y=-x}$	$r_{y=x}$	$r_{\text{y-axis}}$
R_{90°	R_{90°	$r_{\text{y-axis}}$	$r_{\text{x-axis}}$	$r_{y=-x}$	$r_{y=x}$	R_{180°	I	R_{-90°
R_{-90°	R_{-90°	$r_{\text{x-axis}}$	$r_{\text{y-axis}}$	$r_{y=x}$	$r_{y=-x}$	I	R_{180°	R_{90°
R_{180°	R_{180°	$r_{y=-x}$	$r_{y=x}$	$r_{\text{x-axis}}$	$r_{\text{y-axis}}$	R_{-90°	R_{90°	I

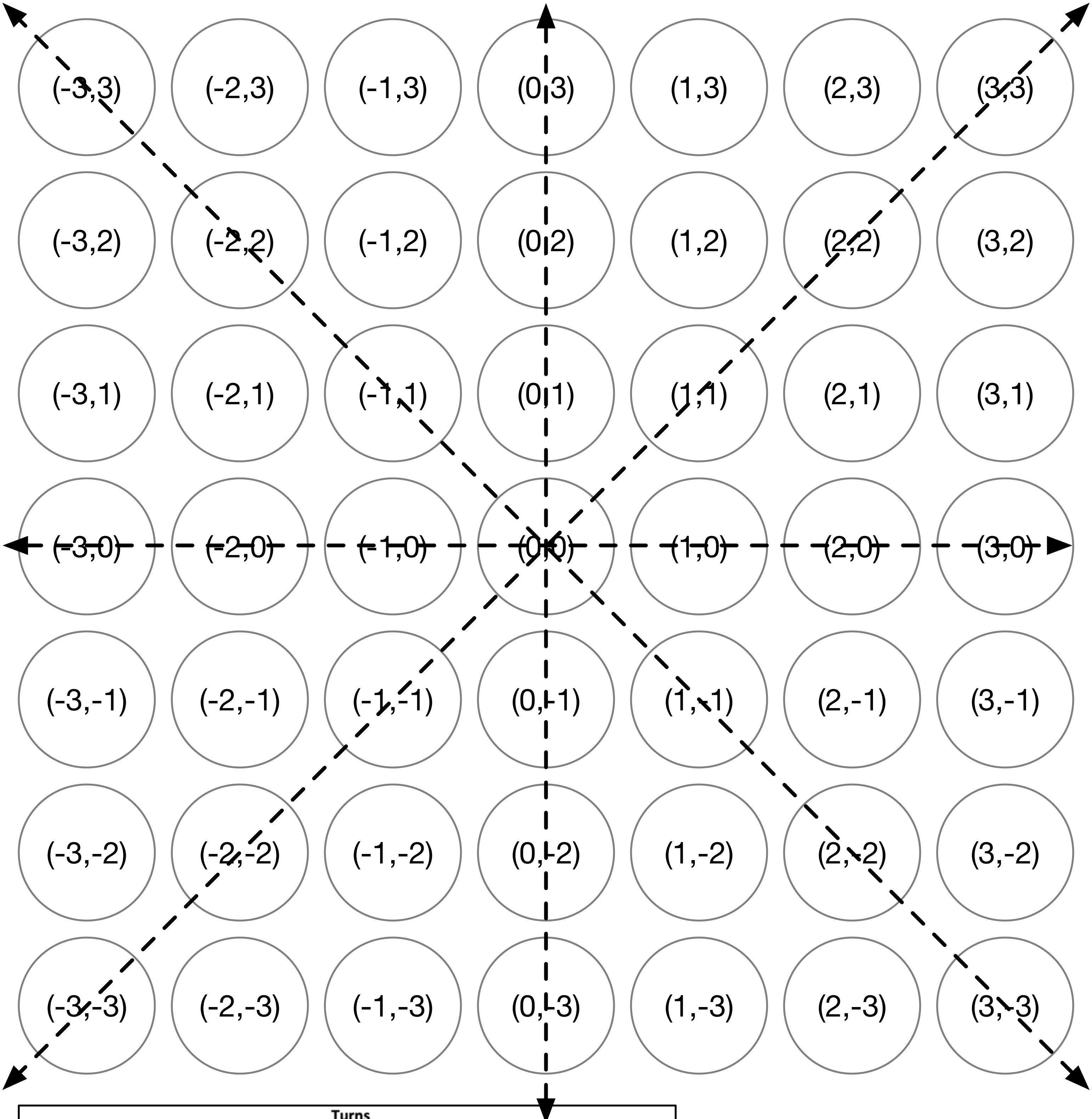
Once you reveal the answers (if you choose to do so), it might be worth discussing the tantalizing symmetry and lack of symmetry informally with your students.

After these activities, your students will be much more familiar with the idea of a composition of transformations, and you may want to let them play the game once more, with the focus cards.

APPENDIX:

Contents:

- Game board with rules printed on the bottom
- The cards
- Follow-up Activity #1 and its answer sheet
- Follow-up Activity #2 and its answer sheet
- Follow-up Activity #3 and its answer sheet
- Slides illustrating the various situations in Part 2

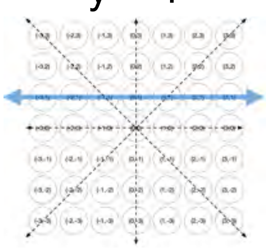
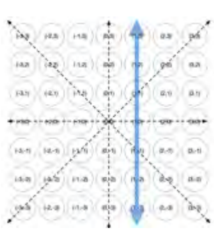
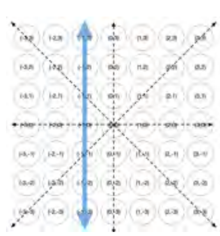
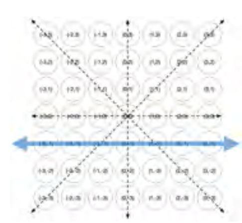


Turns

Players alternate turns. One player must go first; you can let this be decided by the rule "let the player who looked at their reflection most recently go first". Each turn involves the following steps:

- Roll the die. If it lands on 1-3, the player **must** play one non-bonus card. If it lands on 4-6, the player **must** play two non-bonus cards. After each card is played, another is taken from the deck, so one's hand should always contain 4 cards. Played cards go in a discard pile, face up. Then the other player takes a turn.
- Bonus cards can be played at any time, including during the other player's turn, and do not cost a move. The moment one is played, a new card is taken.
- A player cannot have more than 2 bonus cards in their hand. If the player receives a third bonus card, they must discard it and take another card.
- If the deck runs out, shuffle the discard pile. Then that becomes the new deck.

Victory Condition:
A player wins if a card they play pushes the chip off the board during their turn.

ROTATE 90º clockwise around the origin	ROTATE 90º counterclockwise around the origin	ROTATE 180º counterclockwise (but does it matter?) around the origin	IDENTITY Do not change the position
TRANSLATE 1 up ↑	TRANSLATE 1 down ↓	TRANSLATE 1 left ←	TRANSLATE 1 right →
TRANSLATE Roll Die Again 1: 1 unit up 2: 1 unit down 3: 1 unit left 4: 1 unit right 5 or 6: roll again	TRANSLATE Roll Die Again 1: 1 unit up 2: 1 unit down 3: 1 unit left 4: 1 unit right 5 or 6: roll again	BONUS: SWAP (can be played any time; does not cost a move) Swap one of your cards for one of your opponent's cards.	BONUS: DISCARD (can be played any time, does not cost a move) Choose a player. That player discards a card and draws another card.
REFLECT across the line y = 1 	REFLECT across the line x = 1 	REFLECT across the line x = -1 	REFLECT across the line y = -1 

FOCUS

Play three total cards this turn, not including this one

FOCUS

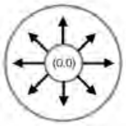
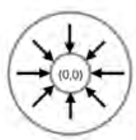
Play a card twice in a row

FOCUS

Play one of your opponent's cards as if it were your own.

FOCUS

Take one extra card from the top of the deck. Your hand now permanently holds one more card.

TRANSLATE 1 up, 1 right 	TRANSLATE 1 up, 1 left 	TRANSLATE 1 down, 1 right 	TRANSLATE 1 down, 1 left 
REFLECT across the y-axis 	REFLECT across the x-axis 	REFLECT across the line y = x 	REFLECT across the line y = -x 
DILATE+ROUND (push out) Multiply both coordinates by 1.5. If you need to round, round so that the chip is further from the origin. 	DILATE+ROUND (pull in) Multiply both coordinates by 0.5. If you need to round, round so that the chip is closer to the origin. 	REFLECT Roll Die Again: 1: across y=1 2: across y=-1 3: across x=1 4: across x=-1 5: across x-axis 6: across y-axis 	ORIGIN Move to (0,0).
BONUS: OTHER WAY (can be played any time; does not cost a move) Play to reverse the direction of a translation immediately after it is played.	BONUS: REROLL (can be played any time; does not cost a move) Play to reroll your die once in any situation	BONUS: EXTEND BOARD (can be played any time, does not cost a move) Place this card outside the game board to extend it so that the chip is not pushed off.	ROTATE Roll Die Again: 1 or 2: 90° clockwise around the origin 3 or 4: 90° counterclockwise around the origin 5 or 6: 180° counterclockwise around the origin

Follow-up Activity #1:

In each space, write what you do to the coordinates (x,y) of the original point to get the coordinates of the transformed point.

REFLECT

across the x-axis

$(x, y) \rightarrow$

REFLECT

across the y-axis

$(x, y) \rightarrow$

REFLECT

across the line y = -x

$(x, y) \rightarrow$

REFLECT

across the line y = x

$(x, y) \rightarrow$

ROTATE

90º clockwise around the origin

$(x, y) \rightarrow$

ROTATE

90º counterclockwise around the origin

$(x, y) \rightarrow$

ROTATE

180º counterclockwise (but does it matter?) around the origin

$(x, y) \rightarrow$

IDENTITY

Do not change the position

$(x, y) \rightarrow$

Follow-up Activity #1
ANSWER KEY

REFLECT	REFLECT	REFLECT	REFLECT
across the x-axis	across the y-axis	across the line y = -x	across the line y = x
$(x, y) \rightarrow$	$(x, y) \rightarrow$	$(x, y) \rightarrow$	$(x, y) \rightarrow$
$(x, -y)$	$(-x, y)$	$(-y, -x)$	(y, x)

ROTATE	ROTATE	ROTATE	IDENTITY
90º clockwise around the origin	90º counterclockwise around the origin	180º counterclockwise (but does it matter?) around the origin	Do not change the position
$(x, y) \rightarrow$	$(x, y) \rightarrow$	$(x, y) \rightarrow$	$(x, y) \rightarrow$
$(y, -x)$	$(-y, x)$	$(-x, -y)$	(x, y)

Activity #2: Express the composition of two transformations as a single transformation

second transformation		first transformation		composition
REFLECT across the y-axis	○	REFLECT across the y-axis	=	
REFLECT across the line y = x	○	REFLECT across the line y = -x	=	
REFLECT across the y-axis	○	REFLECT across the x-axis	=	
ROTATE 90º clockwise around the origin	○	REFLECT across the x-axis	=	
ROTATE 90º clockwise around the origin	○	ROTATE 180º counterclockwise (but does it matter?) around the origin	=	
ROTATE 180º counterclockwise (but does it matter?) around the origin	○	REFLECT across the line y = -x	=	

Activity #2 answer key

second transformation		first transformation		composition
REFLECT across the y-axis	○	REFLECT across the y-axis	=	IDENTITY Do not change the position
REFLECT across the line y = x	○	REFLECT across the line y = -x	=	ROTATE 180º counterclockwise (but does it matter?) around the origin
REFLECT across the y-axis	○	REFLECT across the x-axis	=	ROTATE 180º counterclockwise (but does it matter?) around the origin
ROTATE 90º clockwise around the origin	○	REFLECT across the x-axis	=	REFLECT across the line y = -x
ROTATE 90º clockwise around the origin	○	ROTATE 180º counterclockwise (but does it matter?) around the origin	=	ROTATE 90º counterclockwise around the origin
ROTATE 180º counterclockwise (but does it matter?) around the origin	○	REFLECT across the line y = -x	=	REFLECT across the line y = x

First

Second	I	$r_{y=x}$	$r_{y=-x}$	$r_{\text{y-axis}}$	$r_{\text{x-axis}}$	R_{90°	R_{-90°	R_{180°
I								
$r_{y=x}$								
$r_{y=-x}$								
$r_{\text{y-axis}}$								
$r_{\text{x-axis}}$								
R_{90°								
R_{-90°								
R_{180°								

Follow-up Activity #3:

In each square, write the single transformation that is equivalent to the composition of two transformations: first the one on the top, then the one on the side.

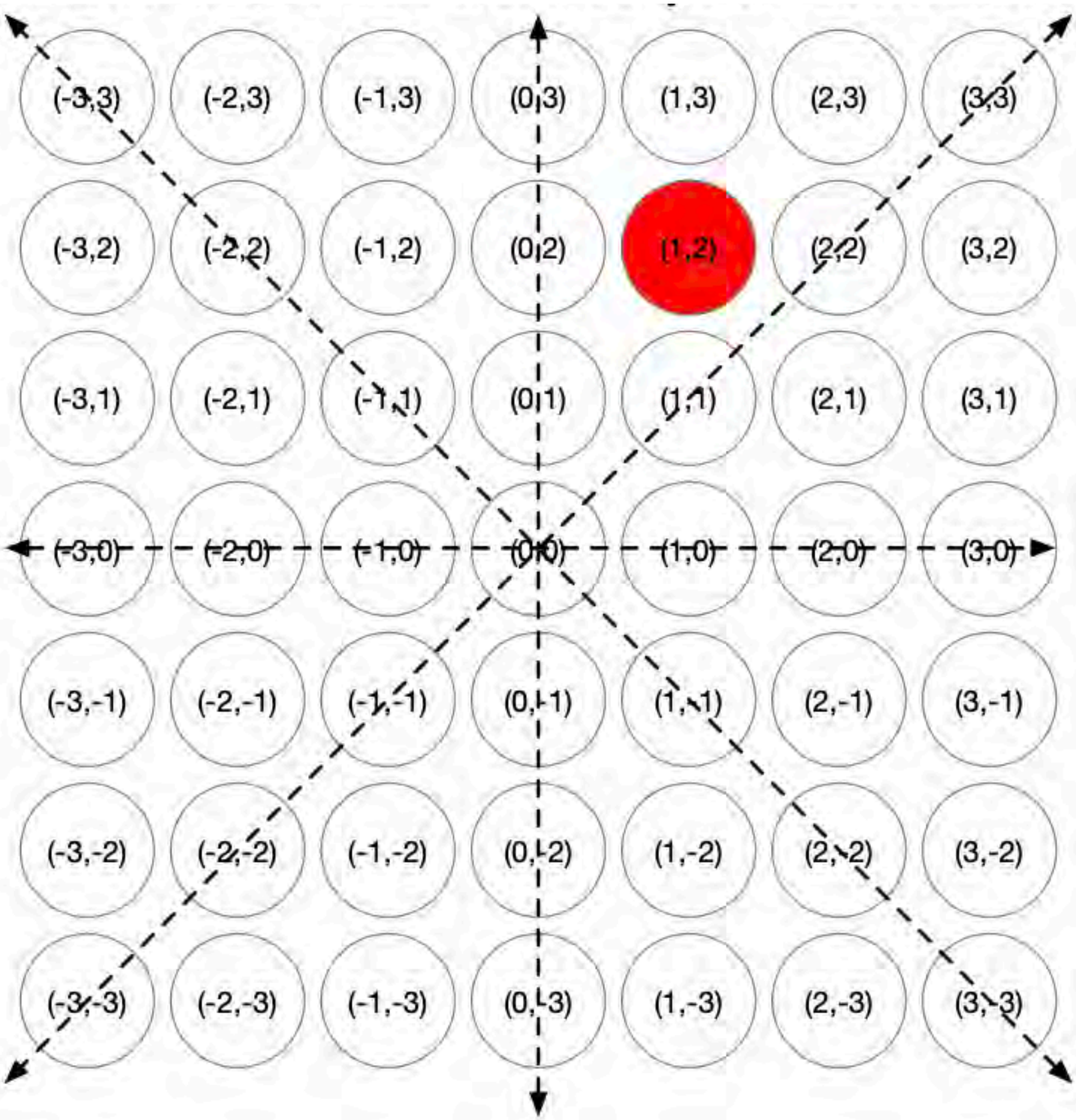
First

Second	I	$r_{y=x}$	$r_{y=-x}$	$r_{\text{y-axis}}$	$r_{\text{x-axis}}$	R_{90°	R_{-90°	R_{180°
I	I	$r_{y=x}$	$r_{y=-x}$	$r_{\text{y-axis}}$	$r_{\text{x-axis}}$	R_{90°	R_{-90°	R_{180°
$r_{y=x}$	$r_{y=x}$	I	R_{180°	R_{-90°	R_{90°	$r_{\text{x-axis}}$	$r_{\text{y-axis}}$	$r_{y=-x}$
$r_{y=-x}$	$r_{y=-x}$	R_{180°	I	R_{90°	R_{-90°	$r_{\text{y-axis}}$	$r_{\text{x-axis}}$	$r_{y=x}$
$r_{\text{y-axis}}$	$r_{\text{y-axis}}$	R_{90°	R_{-90°	I	R_{180°	$r_{y=x}$	$r_{y=-x}$	$r_{\text{x-axis}}$
$r_{\text{x-axis}}$	$r_{\text{x-axis}}$	R_{-90°	R_{90°	R_{180°	I	$r_{y=-x}$	$r_{y=x}$	$r_{\text{y-axis}}$
R_{90°	R_{90°	$r_{\text{y-axis}}$	$r_{\text{x-axis}}$	$r_{y=-x}$	$r_{y=x}$	R_{180°	I	R_{-90°
R_{-90°	R_{-90°	$r_{\text{x-axis}}$	$r_{\text{y-axis}}$	$r_{y=x}$	$r_{y=-x}$	I	R_{180°	R_{90°
R_{180°	R_{180°	$r_{y=-x}$	$r_{y=x}$	$r_{\text{x-axis}}$	$r_{\text{y-axis}}$	R_{-90°	R_{90°	I

Follow-up Activity #3:

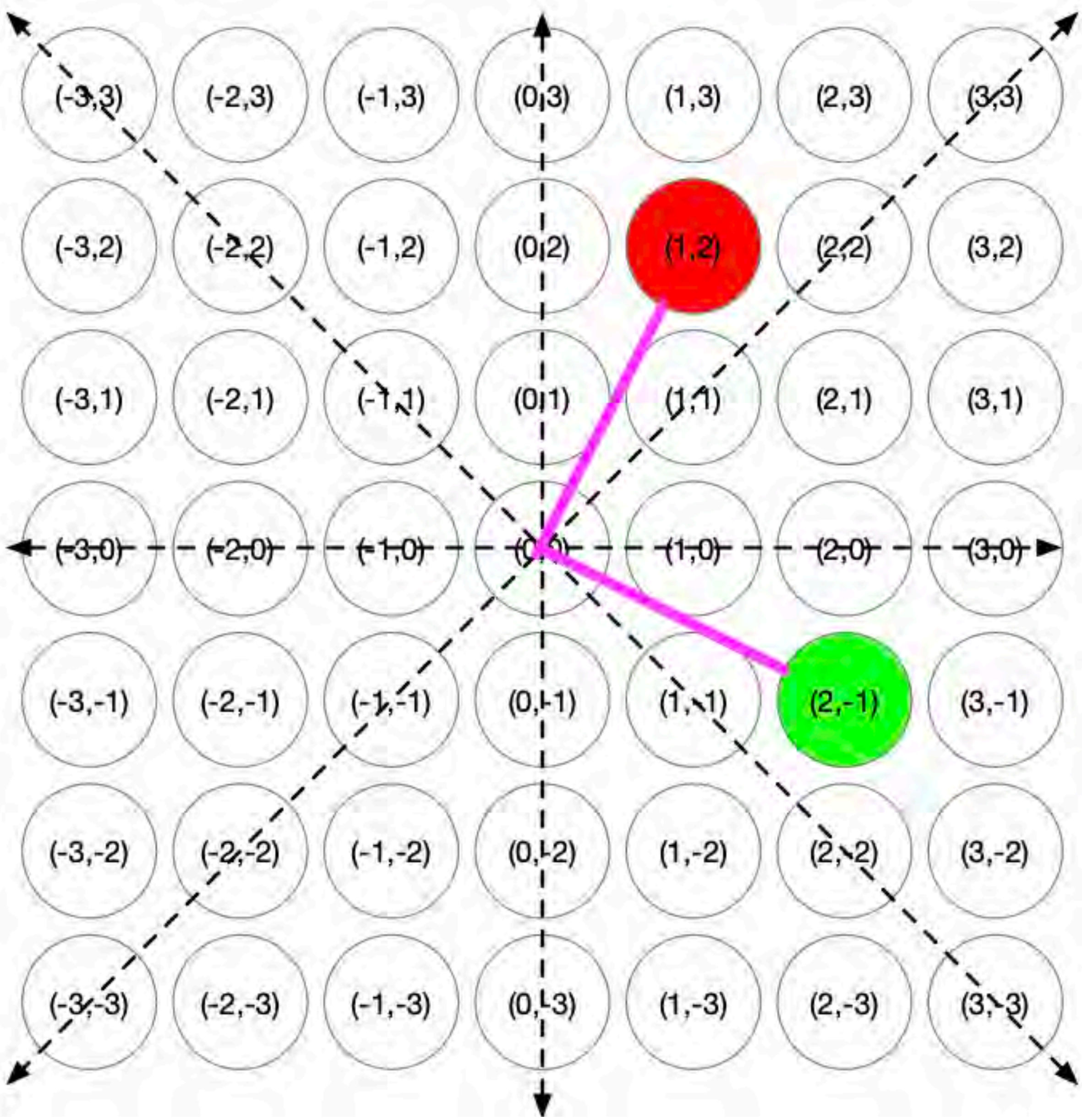
Here is the completed table.

Where will the chip end up?



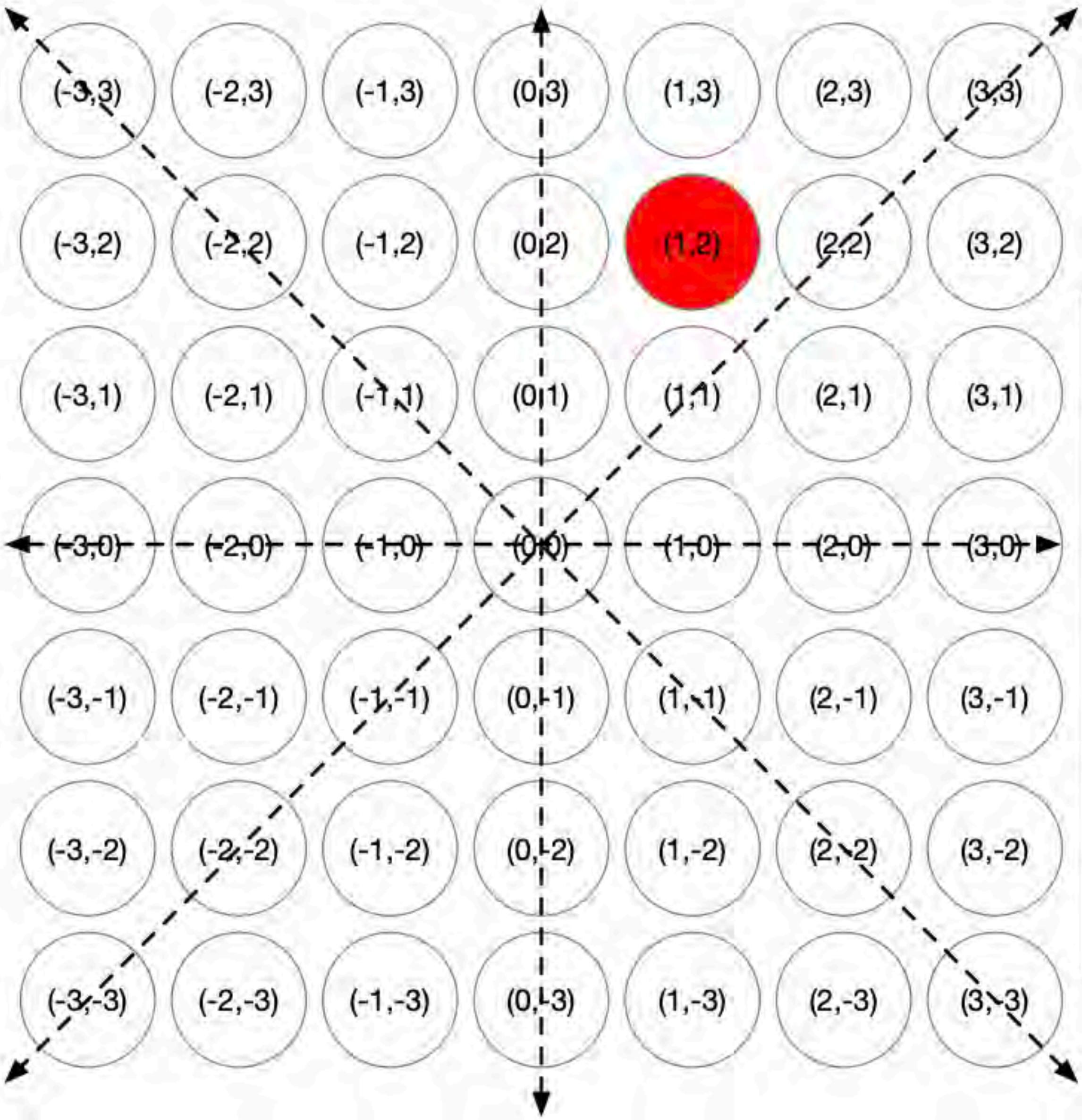
ROTATE

90º clockwise around the origin



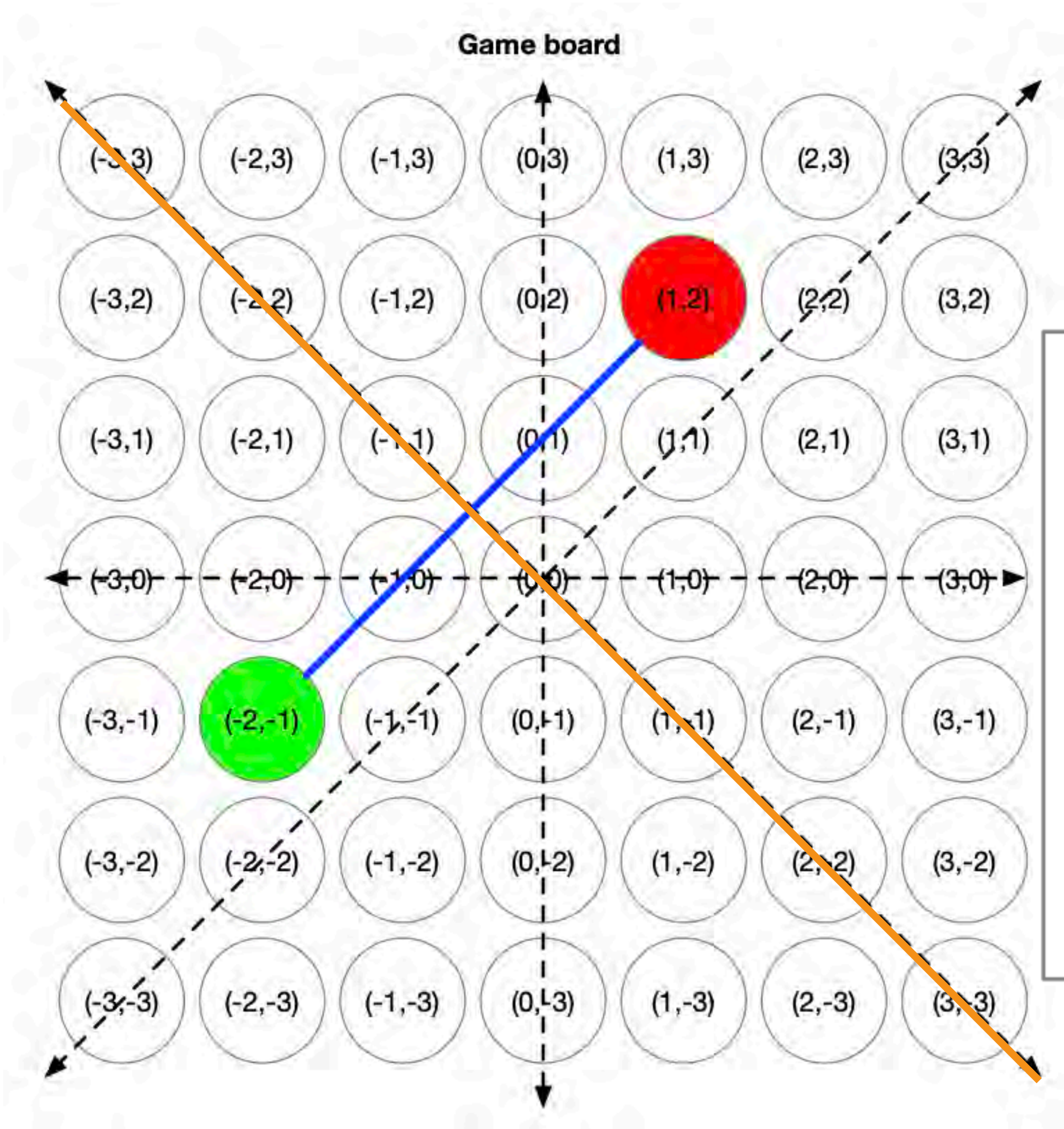
ROTATE

90º clockwise around the origin



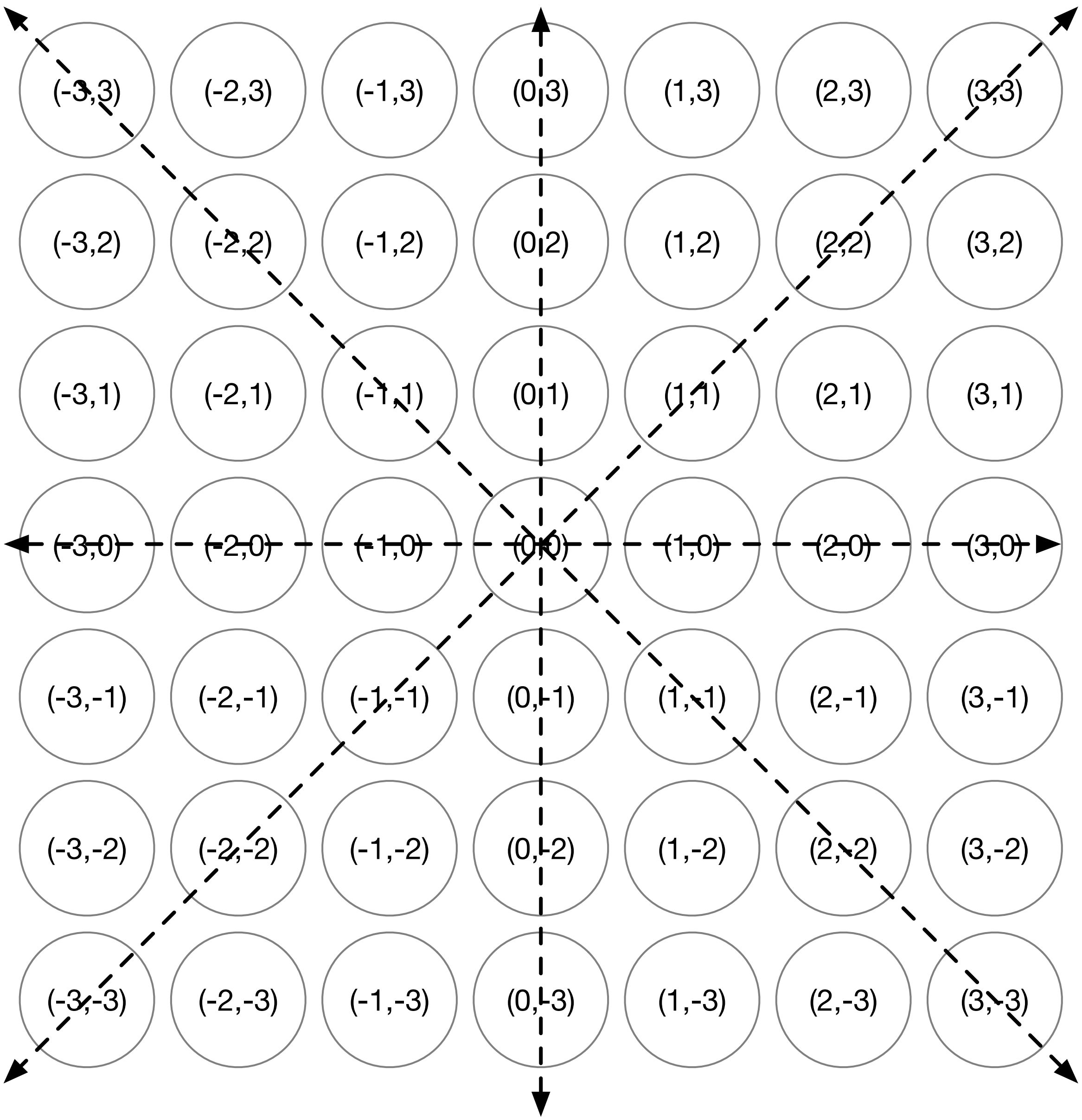
REFLECT

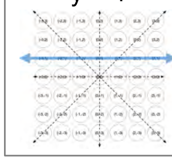
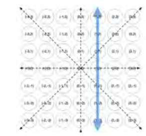
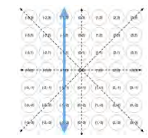
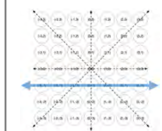
across the line
y = -x

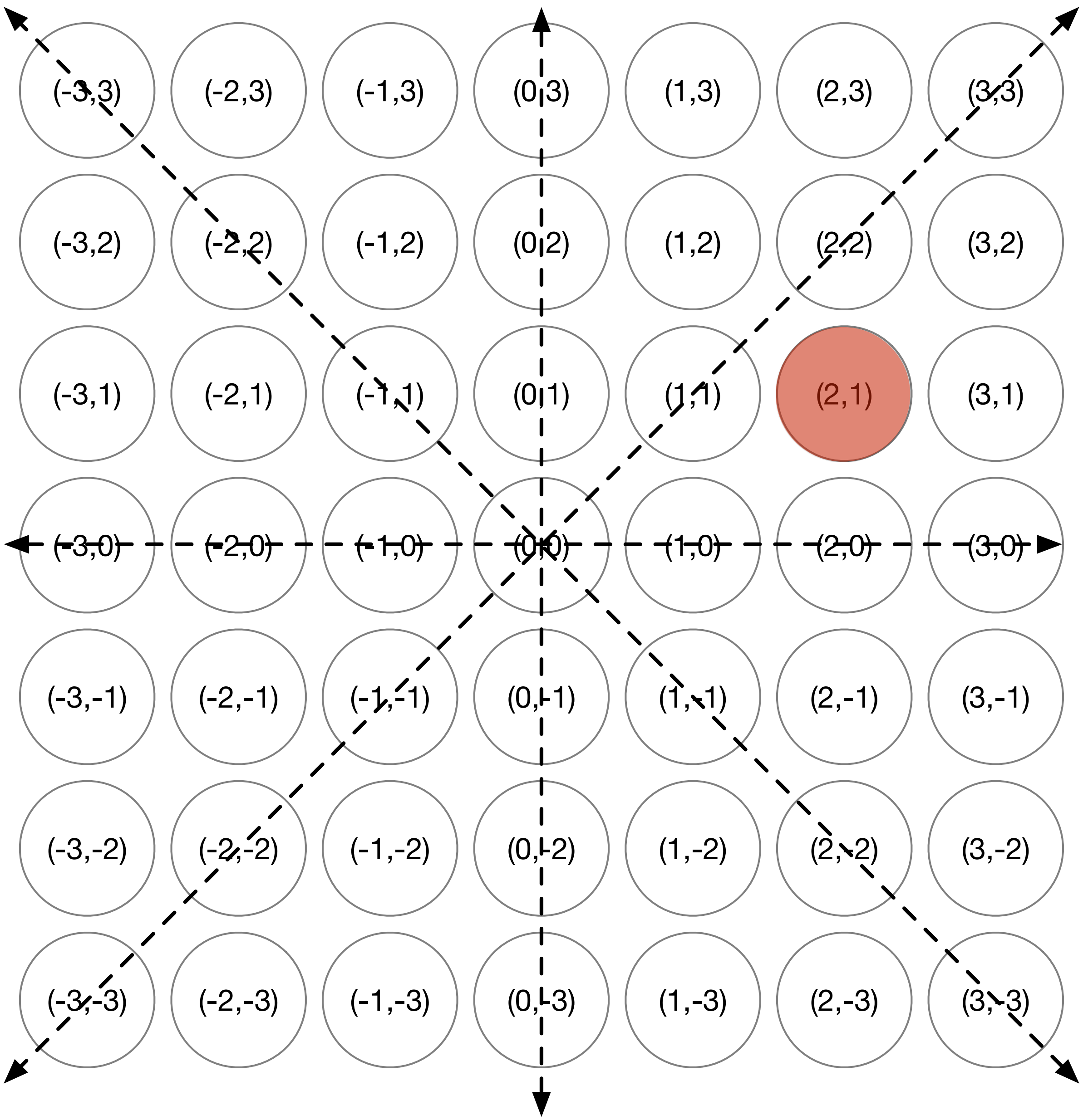


REFLECT

across the line
$y = -x$

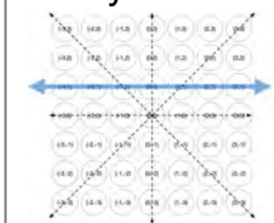


REFLECT across the line y = 1 	REFLECT across the line x = 1 	REFLECT across the line x = -1 	REFLECT across the line y = -1 



REFLECT

across the line
y = 1



ROTATE

90º clockwise
around the
origin

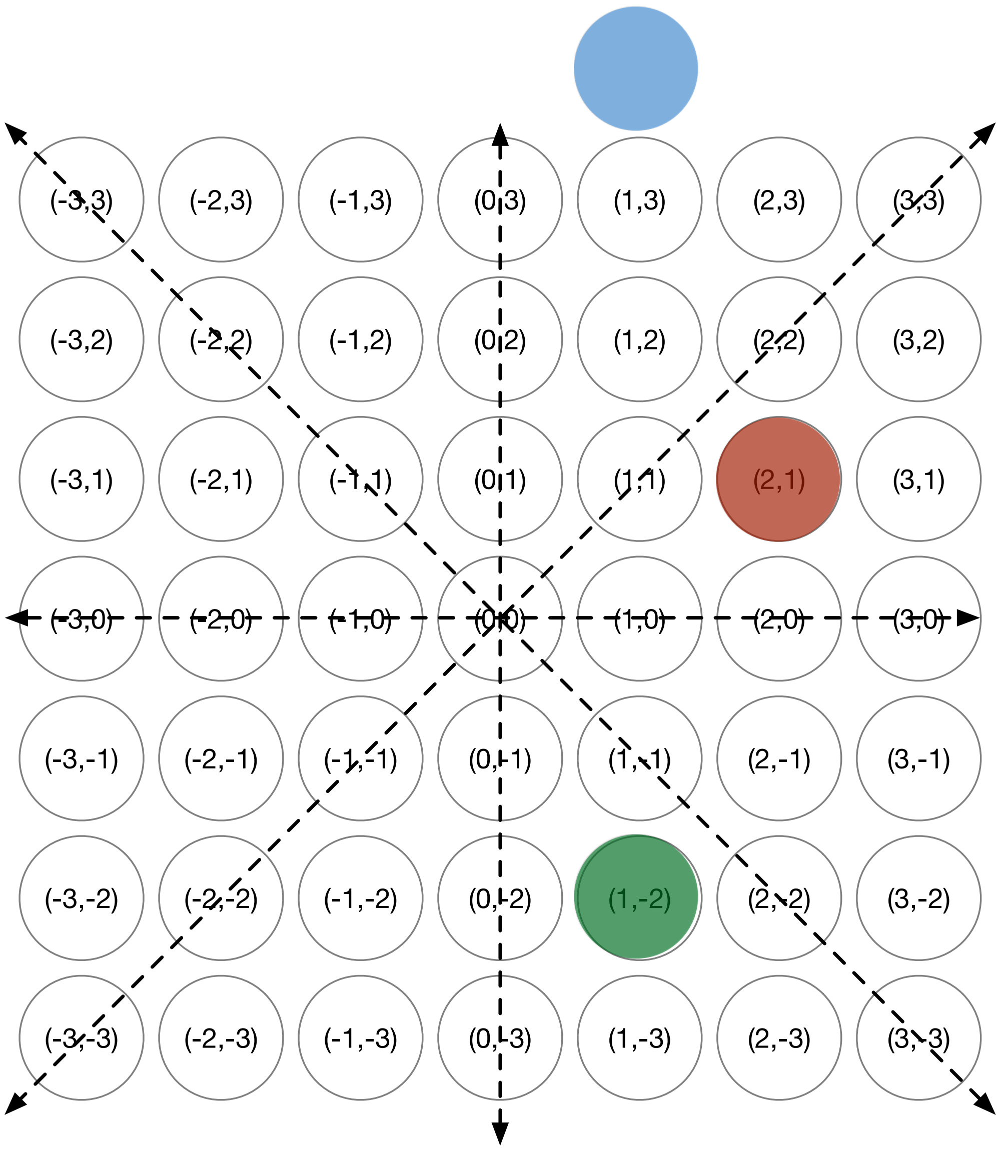
TRANSLATE

1 down, 1 left

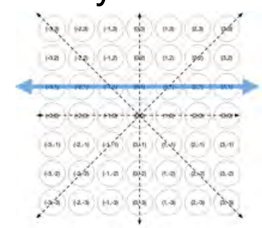
ROTATE

90º
counterclockwise
around the
origin

Can you win in two moves?



REFLECT
SECOND
across the line
y = 1



ROTATE
FIRST
90° clockwise
around the
origin

TRANSLATE

1 down, 1 left

ROTATE

90°
counterclockwise
around the
origin

Can you win in two moves?