MOMATH

NATIONAL MUSEUM OF MATHEMATICS

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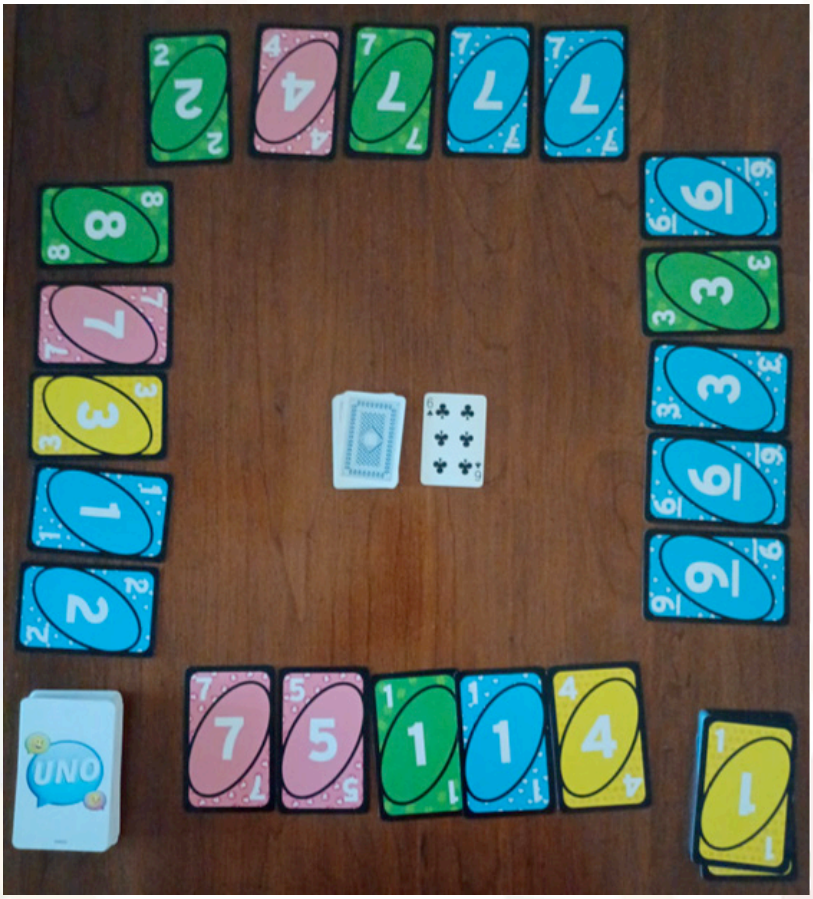
for Innovation and Inspiration in Math Teaching

Divisibilty UNO®

tune in to patterns and turn UP your math power!

Nancy J. Seeds



Grade 6 and more...

Divisibility UNO®

Tune in to patterns and turn UP your math power!

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Overview

What is it?

Divisibility UNO®[1] is a concept game that exercises a broad range of number theory and operations skills in a hands-on and highly integrated fashion, supporting students as they build an important 21st Century math competency: *number literacy.* Though the concept game was originally created to target Grade 6 math goals, it can be successfully used [modified] for younger students and extended to serve all middle school grades.

Why use it?

Have you ever noticed a relative weakness in students' ability to use their acquired skills from prior years when problem solving? Have you been puzzled by math fact recall miscues, such as "6 times 5 is 35" or "12 times 9 is 88" and wondered whether students knew how to assess the reasonableness of their answers? You may be expecting answers to the first example that are, at least, EVEN; to the second, you are anticipating responses that are, at least, greater than 90. While students may demonstrate the ability to use algorithms to compute, are they occasionally unable to develop a strategy to address higher order thinking problems? We – both students and teachers -- have invested several years working to master a diverse set of number concepts and skills. How can we be sure we are also building **number sense**?

How to use it?

One strategy is to teach and apply the different concepts *in association*, rather than discretely (i.e., separately, or sequentially). The more we help students recognize the connectedness of the concepts, the more durable their knowledge and greater their flexibility when problem solving. This concept game models an approach to integrating the following: **patterns in number, primes, composite numbers, divisibility rules, divisor, remainders, multiples, factor trees, factor families, division, addition, subtraction, place value, reading/writing large numbers, LCM, GCF, and computation (including mental math).**

Summary

It is a relatively "simple" game that can build math power and collaboration while having fun! Students engage, discover, create, and communicate mathematically, while you collect fabulous insights into their understanding of number and assess a range of skills in a "low-stress" atmosphere. Why NOT try it?

[1] Uno is a registered trademark of the Mattel Corporation.

Prepare to Engage

The pathways into, through, an out of the activity are varied and flexible. The path you choose can be uniquely individual, considering student needs, productive pacing, and the associations you wish to build upon for follow-on unit topics.

Assuming students already have a solid foundation in divisibility rules and place value, go directly to the game activity (i.e., item #2 in the sequence below). If additional preparation is needed, use the appropriate investigation option in step #1. *For ease in navigation, use the quick links provided for each activity.*

Scope of Activities

1. Use patterns to identify factor families and derive divisibility rules.

> "Developing Divisibility Patterns" is a baseline activity that can be accomplished in about **twenty minutes in a group setting** or developed **through small group hands-on activities over a period of one or two forty-minute sessions**.

2. Practice applying the patterns to problem-solve with larger numbers by playing the game, as students improve math computation skills (i.e., speed & accuracy of mental math and paper & pencil calculations) and utilize place value concepts to maximize scoring opportunities.

> "Divisibility UNO®" is the activity that integrates the concepts into **a hands-on game that can be played in small groups, lasting from twenty to forty minutes per class session** (as time permits).

3. Repeat the activity as often as necessary – and productive – modifying the number of cards used, scaffolding level, and scoring rules.

> "Further Explorations in Divisibility" provides a list of investigation questions that could follow from the application of divisibility patterns, as well as some helpful teacher resources for further study and/or practice.

The desired outcomes for the activity are better flexibility, speed and accuracy when working with numbers, greater independence when problem-solving, and increased student confidence in their math abilities!

Classroom Organization

The activities described in this lesson are intended to support independent action as well as small group and whole group collaboration and discussion. Therefore, flat working surfaces and flexible seating options will support the activities most effectively.

The teacher's roles in the activities are primarily:

- Introduce the lesson with a motivating question or hands-on experience
- Facilitate respectful discussions and reflections

- Model strategies for the activity and determine groupings (i.e., self-select or teacher-select)
- Move through the classroom during the activity, supporting the students with materials and timely questions
- Evaluate individuals through observations and work products

Developing Divisibility Patterns

Many students may have learned about divisibility patterns – or memorized a set of divisibility rules – by the time they enter 6th grade. However, not all may have the same level of understanding or skill. Therefore, a pre-requisite/co-requisite for the activity is to establish a baseline of divisibility patterns, usually developed **by family** to enhance associations and highlight the importance of primes and factors to the patterns existing in composite numbers.

Student Materials

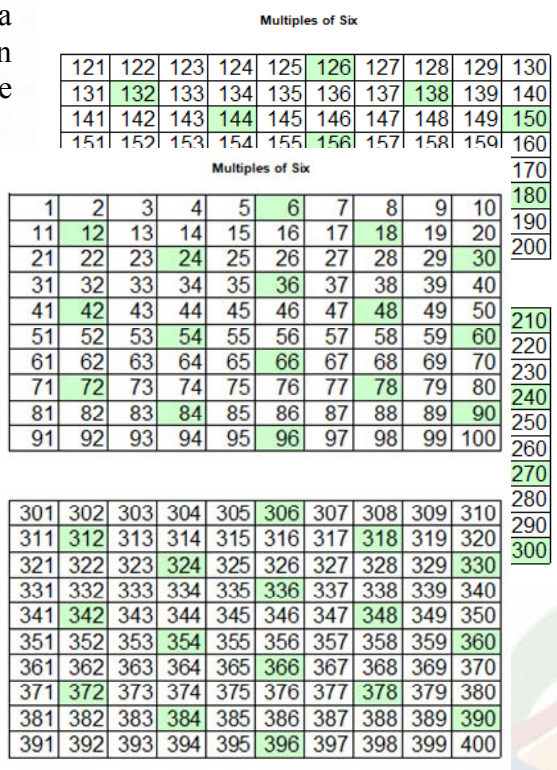
The materials list depends on the type of learning experience most appropriate to the classroom participants and their level of familiarity with the divisibility and pattern recognition.

[Emerging] For **individual hands-on investigation** and group collaboration by family:

- Hundreds Chart blank templates (see Classroom Materials section) - 2 pages each student in a set, one set for each multiple
- Colored pencils or other means of lightly shading
- *Optional (modification) calculator*

[Progressing] For **smaller group investigation** and collaboration by family:

- A 2-page colored set of Hundreds Charts for a given multiple within a family (e.g., 6s as shown at right). A full set of the colored charts are included in the Classroom Materials section.
- *Optional (modification) calculator*



[Mastering] For **whole group *review*** and discussion by family:

- "My Divisibility Rules Reference" sheet for each student ("Divisibility Tests by Family" in Classroom Materials section)
- *Alternatively* use a blank template for students to fill in themselves with meaningful examples (see Classroom Materials section).

The hundreds charts presented on the first page are intentionally non-contiguous, requiring students to develop alternative strategies for finding the first number in the 300s that is divisible by "n" rather than relying on skip counting alone.

Discussion (5-10 minutes)

- What do you know about the multiplies of 2 or 5 or 10 AND what is the connection to divisibility?

- [If they have the blank or colored hundreds charts in front of them…] Can you show where we would find numbers divisible by 2 (or 5 or 10) in these grids. *Hint: They should be "painting stripes" on their paper with their finger flowing down the COLUMNS that are divisible by that number.*

  This is one of the quickest ways to show what they know about the patterns that are visible in strictly the ONES place.

- Do you know, or can you create, a divisibility rule we can reliably use for numbers as they appear in "nature" (i.e., rather than the grid)? E.g., Is 1,705 divisible by 2? What about 5? How do you know? If it is not, what do you expect the remainder to be?

- Do you know, or can you create divisibility rules for other multiples? How about 3 or 6 or 9?

- What if you could determine the divisibility of a LARGE number by 3, 6, 9, 4, 8, and 11 with as much confidence in just a bit longer than it takes for divisibility by 2, 5, and 10. How could that benefit you when problem-solving? E.g., Is 124,875,960 divisible by?

Activity – An Investigation of Divisibility (20-30 minutes)

Divide the class into three or six groups, where each group is responsible for identifying multiples of a given number within the SAME family (e.g., 3, 6, and 9) and for developing/testing a rule that can be used to determine divisibility by their given number. Students within a group share tasks, data, and written conclusions, before sharing them with the class during reflection.

Tip: If students are using the blank charts to shade the multiples, consider whether auditory patterning would be helpful (and less error-prone). For example, one student was comfortable with a "no, no, yes" chant for reliably finding multiples of three; works well for multiples of four, but has declining effectiveness with six, eight, or eleven. All students [eventually] recognize and begin to use the visual patterns revealed in the multiples.

If three groups were formed for the activity, repeat the process using the multiples of four, eight, and eleven (one group for each multiple). If class size enabled six groups to be formed, then each group would work from one of the six factors: 3, 6, 9, and 4, 8, 11.

Note: The elevens pattern will be less obvious to most students, therefore greater creativity and prompting (such as looking at 11*245 as (10+1)*245 or 2450 + 245) may help clue them in to looking at the digits in an alternating pattern.

Reflection (5-10 minutes)

Each group can share their conclusions with the class, respond to questions, and clarify as necessary. Respectful challenges are welcome and generally helpful! Choose from the following examples or create your own…

- Did you find any common patterns among the family groups?
- What can you conclude about the relationship between divisibility and the patterns found within a family?
- What does that suggest about numbers divisible by 12? 15? 18? 44?

Main idea: Composite numbers contain patterns contributed by their underlying prime (and composite) factors. These patterns can be used to simplify the math processes when working with large numbers.

Additional Independent Practice Options

Extend the classroom activity and enhance skills with additional practice opportunities, such as online divisibility games and appropriately leveled worksheet choices listed below.

These can be completed in the classroom and/or at home.

1. Using technology (**untimed** and auto checked) – "VectorKids Divisibility Rules" at *https://vectorkids.com/divisibility.html*

2. Worksheet generators:

   "Divisibility Rules Worksheets" at
   https://www.mathworksheets4kids.com/divisibility-rule.php

   "More Divisibility Rules" at
   https://www.k5learning.com/free-math-worksheets/fifth-grade-5/factoring/divisibility-rules

   "Divisibility Test Division Worksheets" at
   https://www.math-aids.com/Division/Divisibility_Test.html

The Main Tent: Divisibility UNO®

UNO® cards were selected rather than regular cards because they contain the number strictly as a digit, making the connection to constructing a multi-digit number out of cards much clearer and concrete.

The game has been extended, formalized, over time to include more divisibility rules, and higher degrees of difficulty (adding more digits and scoring rules changes). *I am sure that teachers can find even more creative ways to modify this lesson plan and game to meet their own class needs.*

Student Materials

- Deck of UNO® cards (with "special" cards removed); one set for every 4 students will be sufficient.
- One playing card each from 2-10 plus ace (as "11") from a common deck of cards, which is used for random "divisor" selection. (*Rolling two six-sided dice may be substituted for the random divisor selection*.)
- Individual sheet of paper for scorekeeping: lined, grid, or blank (as appropriate for student need)
- "Divisibility UNO®" game rules (see Classroom Materials section).
- *Optional* – a Divisibility Rules reference sheet (see Classroom Materials section).

Discussion (5-10 Minutes)

State that you would like to teach them a new game, but it requires the ability to efficiently identify whether a number is going to be divisible by another number...

- Tell me everything you know about the number 816. *(Answers will vary, of course!)*
- What if we make the number larger by adding digits? Can you suggest some digits we can add to the beginning of the number? (*Two to four additional digits would be sufficient here.*)
- Tell me everything you know about this larger number.

When the game activity is repeated, Problem of the Day (POD) challenges can take the place of the discussion prompts above. POD examples:
"Find the largest 4-digit number that will be divisible by 6."
“Find the smallest 5-digit number that is divisible by 11.”
"Can you find a 5-digit number that is divisible by both 4 and 9? What other number(s) must that 5-digit number be divisible by?”

Activity – Divisibility UNO® Concept Game (20-35 minutes)

Game details are provided in the Classroom Materials section, titled “Divisibility UNO® Game Rules.”

Game Option 1: Model the game with **4 cards** and a **negotiated set of divisors** in groups of 4-5 with one set of written divisibility rules for each group. (One set of rules sets up an environment of cooperation and communication.) *Consider omitting the scoring component on the first time with the game, so that students can focus on efficiently creating 4-digit numbers that pass the divisibility*

test. Post-activity independent practice: Study divisibility rules and use them to finish a sample divisibility worksheet with 3- and 4-digit numbers.

Game Option 2: Play the game with **4 cards** and **full set of divisors** and one set of written divisibility rules, adding scoring component (if initially omitted). First player to 100,000 wins. Post-activity independent practice: Study divisibility rules. Try completing the worksheet of 4- and 5-digit numbers without referring to the divisibility reference sheet. How about teaching someone at home how to play Divisibility UNO®?

Game Option 3: Play the game to 1 million with **5 cards without use of a reference sheet**, keeping a running score. Students can receive pattern reminders from others in the group, when necessary. First player to 1,000,000 wins! (Modification: If player has forgotten a rule, they can "pass" or ask another player for assistance. If the other player helps, then the number eventually played is split 2 ways--half to the assistant and half to the requestor. *Good mental math exercise to divide by two!*)

Play additional games of **5-card** Divisibility Uno®, where players begin with 1,000,000 and SUBTRACT**.** First player to zero wins! Post-activity practice: Augment games with worksheets of divisibility problems and factoring skills.

Differentiated Design Elements

1. (Scaffolding) Student cards left face-up on the table, so that everyone can observe, suggest, question, and solve with support.

2. (Memory) Students can be granted the use of their OWN set of divisibility rules to refer to at any time during the game.

3. (Organization) Graphic organizer presentation of rules, rather than text, list. One visual example is provided in the Classroom Materials section, titled “Divisibility Tests by Family.”

4. (Extra challenge: 7s divisibility) If trial & error/mental math for 7s test is taking too much time and/or is too difficult for students to master, allow use of calculators ONLY for divisor=7 or play without a 7 in the divisor pile.

5. (Game rule negotiations) Rules of play can be changed by teacher or students to increase difficulty and strategies--such as 6-card, subtraction vs. addition, "challenge and capture" another's points, time penalties, using the Jack/Queen/King (for divisors of 12/15/18, respectively).

Students seem to love negotiating rules!

Formative Assessment Options

1. Collect and check student score sheet(s) - divisibility as well as addition (or subtraction) with regrouping.
2. Assess quality, speed, independence of play and strategies through observation of the groups as they play.
3. Written responses to post-activity independent practice assignments and Problem of the Day (POD) challenge(s) such as "Find the largest 6-digit number that will be divisible by 9." “Find the smallest 5-digit number that is divisible by 4.”

Summative Assessment Options

1. Sufficient accuracy (>=75 or 80%) on post-test about divisibility, factors, and remainders.
2. "Championship Divisibility Uno®" tournament; play as individuals or as small teams playing other teams in a table rotation scheme.
3. Apply the concepts and skills in novel situations to demonstrate mastery of skills and effective higher-order thinking. One such example is provided in the Classroom Materials section, titled "Problem-Solving with Divisibility" (and includes answer key).

Reflection and Closure

1. Post-activity challenge: "Find a PERFECT Divisibility Uno® 5-digit number." A perfect number would be one divisible by 2,3,4,5,6,7,8,9,10 AND 11!

2. Follow-up question: "Is there another PERFECT 5-card number possible? If so, how many others are there in all?" (*FYI: Factor trees can be used to find the LCM of all these divisors; any 5-digit multiple of this LCM will also be a perfect hand*.) Will students discover others using alternative methods?

3. Extra challenge questions: "Is there a 5-card hand that would make it impossible to play anything other than the same as the divisor (i.e., divisor times 1)?" and "How many of these IMPOSSIBLE hands can YOU find?"

4. Construct a concept map (with student input) and explicitly/concretely make the "connect" to the next lesson on another number theory topic (such as primes/composites, prime factorization, exponents, GCF, LCM, etc.) or as the beginning of a unit on "Parts of a Whole" (fraction ←→ decimal).

Figure 1: Photo of a small group playing Divisibility Uno with 5-card hands racing to 1 million!
(Notice that a soroban was provided instead of a calculator.)

Author Summary

The activity has been effective for use as a beginning of year assessment tool for the older grades (7 and 8), enabling identification of computational, memorization, and/or conceptual strengths and weaknesses that are observed within a group.

Students chose to share the activity with special visitors one recent year. The adults were surprised at how much more quickly students were able to arrive at solutions than they were—even if they had assumed the students to not be as "fluent" in mechanical computation as the adults themselves. Most needed the assistance of the students to find a number in their hand to play. One was overheard to say, "Why didn't they teach us this when WE were in school?" Many visitors noticed that it was a very different way to look at numbers, draw conclusions, and solve problems.

Virtually all saw the patterns and strategies the students were employing as superior to their own.

Further Explorations in Divisibility

There are many connections that can be made, both backward and forward, to the activities presented as part of the Divisibility UNO® lesson activities in this packet. A short list of topics and educator resources can be found below.

- Investigate how divisibility patterns can be used to identify REMAINDERS for a number.
- Try using divisibility patterns to SIMPLIFY a division problem with 2- or 3-digit divisors. (See one example, "Patterns as Process," included in the Classroom Materials.)
- Use the Sieve of Eratosthenes to identify the PRIMES to 100. Recalling them is very beneficial to all factoring tasks in elementary and high school mathematics! (See the "Shape of Primes to 100" in the Classroom Materials, which has been found to be helpful to commit the prime numbers to memory.)
- Use prime numbers and factor trees to develop prime factorizations of 2- and 3-digit numbers.
- Explore a process for using factor trees and Venn diagrams to identify the GCF and compute the LCM of two numbers. One investigation example uses an application, eNLVM, to scaffold students through the process. (See image below.)

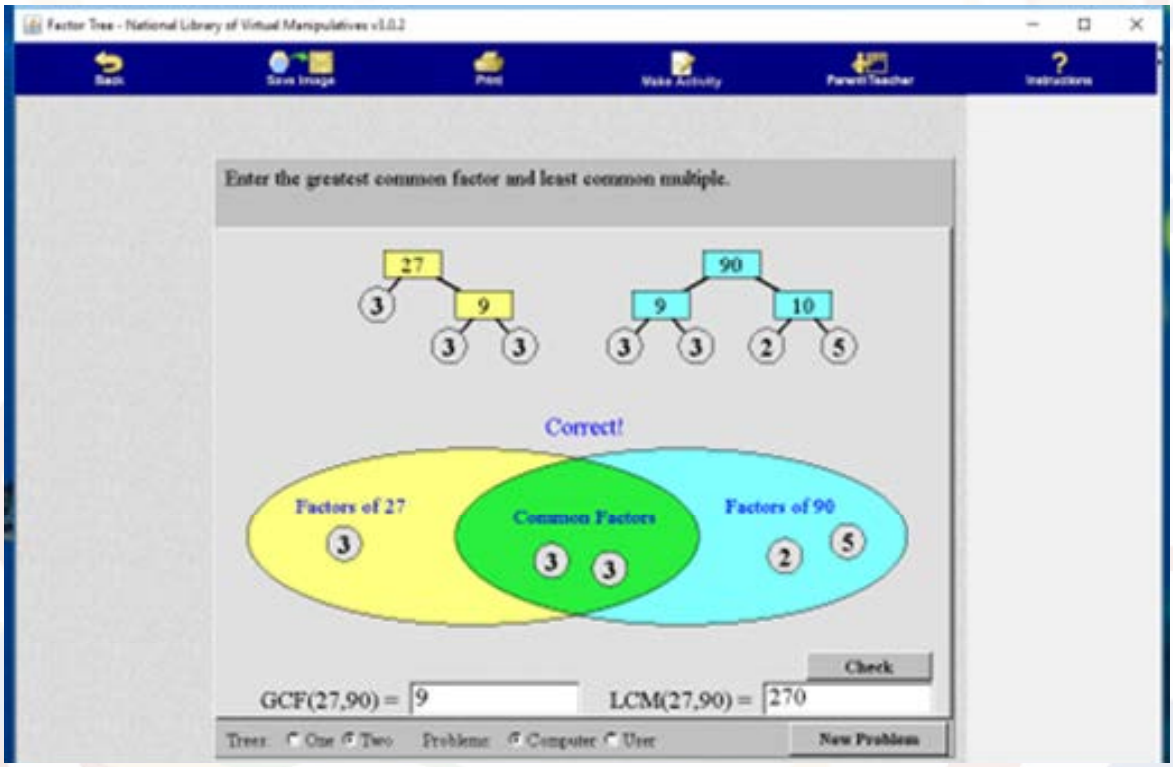


Figure 2: A screen shot taken from the lab activity using the National Library of Virtual Manipulatives "Factor Tree" desktop application. *Note: A Java version is available at http://nlvm.usu.edu/en/nav/vlibrary.html but is not supported on many browsers.*

Common Core and P21

Instructional time is a finite – and precious – resource. Common Core standards (CCSS) and the Partnership for 21st Century Learning (P21.org) helped guide both content and practice for students in these activities. *Note: P21 has since been incorporated into the BatelleforKids Network (www.battelleforkids.org).*

Specifically, the following Common Core standards are addressed:

- Compute fluently with multi-digit numbers and find common factors and multiples. (CCSS.MATH.CONTENT.6.NS.B.2, CCSS.MATH.CONTENT.6.NS.B.3, CCSS.MATH.CONTENT.6.NS.B.4)
- All eight of the mathematical practice standards for the grade.

The P21 Common Core Toolkit aligns the Common Core Standards and the P21 Framework and can be accessed at *https://www.battelleforkids.org/networks/p21*. Of particular P21 focus were:

- Learning and Innovation Skills (summarized on pages 2 and 5 of the document "21st Century Skills Map," which can be found at https://files.eric.ed.gov/fulltext/ED543032.pdf, published Jan 2011).

A relevant page 2 excerpt from the document is included here for clarification.

Learning and Innovation Skills

Learning and innovation skills are increasingly recognized as those that distinguish students who are prepared for more complex life and work environments in the 21st century from those who are not. A focus on the 4C's of Creativity, Critical thinking, Communication and Collaboration is essential to prepare students for the future.

Creativity and Innovation: Students use a wide range of techniques to create new and worthwhile ideas, elaborate, refine, analyze and evaluate their own ideas in order to improve and maximize creative efforts, and demonstrate originality and inventiveness, in both an individual as well as group settings.

Critical Thinking and Problem Solving: Students reason effectively, use systems thinking and understand how parts of a whole interact with each other. They make judgments, decisions and solve problems in both conventional and innovative ways.

Communication and Collaboration: Students know how to articulate thoughts and ideas effectively using oral, written and nonverbal communication. They listen effectively to decipher meaning, such as knowledge, values, attitudes and intentions, and use communication for a wide range of purposes in diverse teams and environments.

Classroom Materials

Materials to selectively print out and use in the classroom are included on the following pages. The documents appear in the order in which they are introduced in the lessons.

1	2	3	4	5	6	7	8	9	10
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Multiples of Three

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Multiples of Three

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Multiples of Six

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Multiples of Six

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361	362	363	364	365	366	367	368	369	370
371	372	373	374	375	376	377	378	379	380
381	382	383	384	385	386	387	388	389	390
391	392	393	394	395	396	397	398	399	400

Multiples of Nine

101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150
151	152	153	154	155	156	157	158	159	160
161	162	163	164	165	166	167	168	169	170
171	172	173	174	175	176	177	178	179	180
181	182	183	184	185	186	187	188	189	190
191	192	193	194	195	196	197	198	199	200

201	202	203	204	205	206	207	208	209	210
211	212	213	214	215	216	217	218	219	220
221	222	223	224	225	226	227	228	229	230
231	232	233	234	235	236	237	238	239	240
241	242	243	244	245	246	247	248	249	250
251	252	253	254	255	256	257	258	259	260
261	262	263	264	265	266	267	268	269	270
271	272	273	274	275	276	277	278	279	280
281	282	283	284	285	286	287	288	289	290
291	292	293	294	295	296	297	298	299	300

Multiples of Nine

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

301	302	303	304	305	306	307	308	309	310
311	312	313	314	315	316	317	318	319	320
321	322	323	324	325	326	327	328	329	330
331	332	333	334	335	336	337	338	339	340
341	342	343	344	345	346	347	348	349	350
351	352	353	354	355	356	357	358	359	360
361	362	363	364	365	366	367	368	369	370
371	372	373	374	375	376	377	378	379	380
381	382	383	384	385	386	387	388	389	390
391	392	393	394	395	396	397	398	399	400

Multiples of Four

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

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311	312	313	314	315	316	317	318	319	320
321	322	323	324	325	326	327	328	329	330
331	332	333	334	335	336	337	338	339	340
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371	372	373	374	375	376	377	378	379	380
381	382	383	384	385	386	387	388	389	390
391	392	393	394	395	396	397	398	399	400

Multiples of Four

101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150
151	152	153	154	155	156	157	158	159	160
161	162	163	164	165	166	167	168	169	170
171	172	173	174	175	176	177	178	179	180
181	182	183	184	185	186	187	188	189	190
191	192	193	194	195	196	197	198	199	200

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211	212	213	214	215	216	217	218	219	220
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231	232	233	234	235	236	237	238	239	240
241	242	243	244	245	246	247	248	249	250
251	252	253	254	255	256	257	258	259	260
261	262	263	264	265	266	267	268	269	270
271	272	273	274	275	276	277	278	279	280
281	282	283	284	285	286	287	288	289	290
291	292	293	294	295	296	297	298	299	300

Multiples of Eight

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

301	302	303	304	305	306	307	308	309	310
311	312	313	314	315	316	317	318	319	320
321	322	323	324	325	326	327	328	329	330
331	332	333	334	335	336	337	338	339	340
341	342	343	344	345	346	347	348	349	350
351	352	353	354	355	356	357	358	359	360
361	362	363	364	365	366	367	368	369	370
371	372	373	374	375	376	377	378	379	380
381	382	383	384	385	386	387	388	389	390
391	392	393	394	395	396	397	398	399	400

Multiples of Eight

101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150
151	152	153	154	155	156	157	158	159	160
161	162	163	164	165	166	167	168	169	170
171	172	173	174	175	176	177	178	179	180
181	182	183	184	185	186	187	188	189	190
191	192	193	194	195	196	197	198	199	200

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211	212	213	214	215	216	217	218	219	220
221	222	223	224	225	226	227	228	229	230
231	232	233	234	235	236	237	238	239	240
241	242	243	244	245	246	247	248	249	250
251	252	253	254	255	256	257	258	259	260
261	262	263	264	265	266	267	268	269	270
271	272	273	274	275	276	277	278	279	280
281	282	283	284	285	286	287	288	289	290
291	292	293	294	295	296	297	298	299	300

Multiples of Eleven

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

301	302	303	304	305	306	307	308	309	310
311	312	313	314	315	316	317	318	319	320
321	322	323	324	325	326	327	328	329	330
331	332	333	334	335	336	337	338	339	340
341	342	343	344	345	346	347	348	349	350
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361	362	363	364	365	366	367	368	369	370
371	372	373	374	375	376	377	378	379	380
381	382	383	384	385	386	387	388	389	390
391	392	393	394	395	396	397	398	399	400

Multiples of Eleven

101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150
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161	162	163	164	165	166	167	168	169	170
171	172	173	174	175	176	177	178	179	180
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191	192	193	194	195	196	197	198	199	200

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221	222	223	224	225	226	227	228	229	230
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241	242	243	244	245	246	247	248	249	250
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261	262	263	264	265	266	267	268	269	270
271	272	273	274	275	276	277	278	279	280
281	282	283	284	285	286	287	288	289	290
291	292	293	294	295	296	297	298	299	300

Multiples of Seven

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

301	302	303	304	305	306	307	308	309	310
311	312	313	314	315	316	317	318	319	320
321	322	323	324	325	326	327	328	329	330
331	332	333	334	335	336	337	338	339	340
341	342	343	344	345	346	347	348	349	350
351	352	353	354	355	356	357	358	359	360
361	362	363	364	365	366	367	368	369	370
371	372	373	374	375	376	377	378	379	380
381	382	383	384	385	386	387	388	389	390
391	392	393	394	395	396	397	398	399	400

My Divisibility Rules Reference

Divisibility Test[1]	Examples
A number is **divisible by 2** if the last digit is a number divisible by two (i.e., EVEN: 2, 4, 6, 8 or 10).	168 **is** divisible by 2 since the last digit is 8. 2,107 **is not** divisible by 2 since the last digit is 7.
A number is **divisible by 4** if the number formed by the last two digits is divisible by 4.	316 **is** divisible by 4 since 16 is divisible by 4. 168 **is** also divisible by 4 since 68 is divisible by 4. 426 **is not** divisible by 4 since 26 isn't divisible by 4.
A number is **divisible by 8** if the number formed by the last three digits is divisible by 8.	7,120 **is** divisible by 8 since 120 is divisible by 8. 45,168 **is** divisible by 8 since 168 is divisible by 8. 1,220 **is not** divisible by 8 since 220 isn't divisible by 8.
A number is **divisible by 3** if the sum of the digits is divisible by 3.	168 **is** divisible by 3 since the sum of the digits is 15 (1+6+8=15), and 15 is divisible by 3. 2,107 **is not** divisible by 3 since the sum of the digits is 10, which isn't divisible by 3.
A number is **divisible by 6** if it is divisible by 2 **AND** it is divisible by 3.	168 **is** divisible by 6 since it is divisible by 2 **AND** it is divisible by 3. 212 **is not** divisible by 6 since it is divisible by 2 but is not divisible by 3.
A number is **divisible by 9** if the sum of the digits is divisible by 9.	549 **is** divisible by 9 since the sum of the digits is 18 (5+4+9=18), and 18 is divisible by 9. 1,236 **is not** divisible by 9 since the sum of the digits is 12, which isn't divisible by 9.
A number is **divisible by 5** if the last digit is either 0 or 5.	195 **is** divisible by 5 since the last digit is 5. 2,107 **is not** divisible by 5 since the last digit is neither 5 nor 0.
A number is **divisible by 10** if the last digit is 0.	1,470 **is** divisible by 10 since the last digit is 0. 762 **is not** divisible by 10 since the last digit isn't 0.
A number is **divisible by 11** if the ODD digits sum minus the EVEN digits sum is a multiple of eleven. (aka "UP/DOWN" rule)	96,261 **is** divisible by 11 since the sum of the odd digits (9+2+1) MINUS the sum of the even digits (6+6) is ***0*** or ***a multiple of eleven***. 96,621 **is not** divisible by 11 since (9+6+1) – (6+2) = 8.

[1] All divisibility strategies for 7 are trial & error with too many steps, so it is not included as part of this reference.

My Divisibility Rules Reference

Divisibility Test[1]	Examples
A number is **divisible by 2** if	
A number is **divisible by 5** if	
A number is **divisible by 10** if	
A number is **divisible by 3** if	
A number is **divisible by 6** if	
A number is **divisible by 9** if	
A number is **divisible by 4** if	
A number is **divisible by 8** if	
A number is **divisible by 11** if	

[1] Divisibility strategies for 7 are **trial & error** with no pattern (and too many steps), so it is not included as part of this reference.

Divisibility UNO® Game Rules

A math concept game designed by NJ Seeds to address student needs observed in the classroom. (*Note: UNO® is a registered trademark of the Mattel Corporation*.)

Overview - Number Theory & Divisibility Concept Game

- Concepts involved: Patterns in number, primes, composite numbers, divisibility rules, divisor, remainders, multiples, factor trees, factor families, division, addition, subtraction, place value, large numbers, LCM, computation, regrouping, mental math

- Object of the game: Designed to be played from 1 to "some big number of players". Each player will be trying to make the largest number possible from their hand that is divisible by "n" during each round of play.

- Materials needed:
  1. **Player Deck** - One (or more) deck(s) of UNO® cards with "special" cards removed. *Use more than one deck of cards if you have more than 4 players; it will make game play proceed faster by avoiding shuffles.*

  2. **Divisor Deck** - One suit of playing cards (Ace through 10 only). *Note: Rolling two six-sided dice could substitute for the Divisor Deck.*

  3. **Score Sheet** – lined, graph, or plain paper for maintaining score and/or use as scratch paper.

Number of Players

This game can be played with **as few as one** and as many as feasible (especially if many Uno decks are available). Collaborative groups of 4-5 are usually best for the learning phase of the process. Whole classes can participate in a Divisibility UNO® Tournament, if enough materials are available.

Preparing to Play

Before the Game: Students should be familiar with patterns in number and number families. Exploration of these patterns should lead to the development of divisibility "rules" that can be applied consistently to determine whether any number "x" is divisible by "n". *(A sample sheet of divisibility rules is included in the Classroom Materials.)*

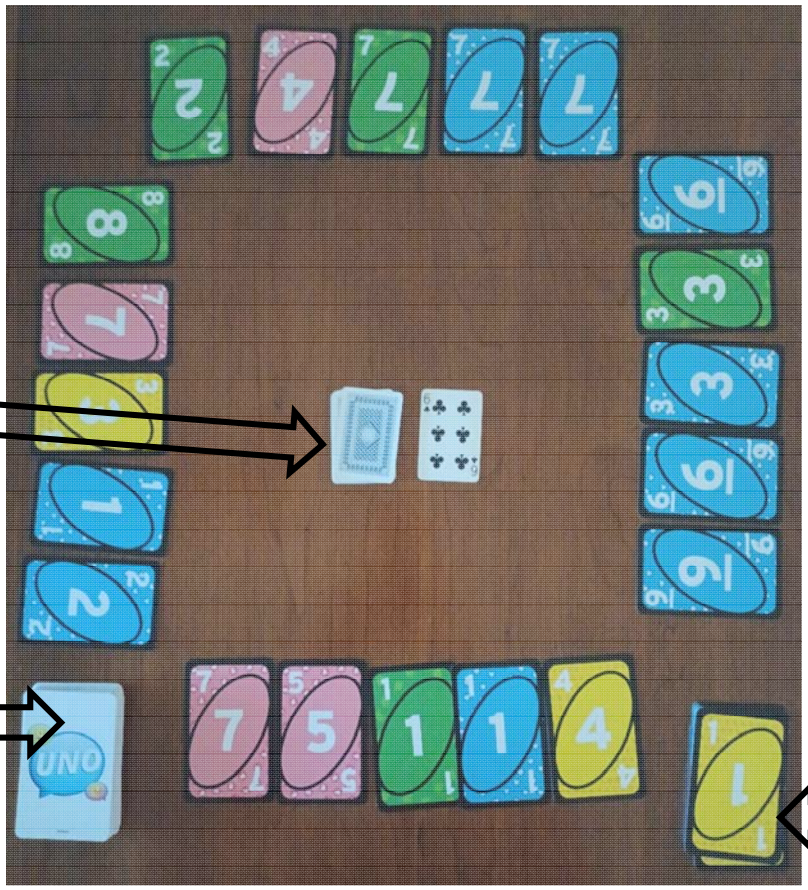
Setup: Determine—in advance—how many cards each person will get, i.e., 4-digit, 5-digit, or more. Shuffle the Uno® deck and deal each player the agreed number of cards, FACE UP. Place the remaining cards off to the side to be used in the next round. (**Cards in play should always be visible to all**; particularly valuable for students learning from each other.)

If a Divisor Deck is being used, shuffle the cards, and place them FACE DOWN in the middle of the table, so that all can see. *If dice are used, players take turns rolling for the divisor each round.*

An image of a 4-player game in progress in included below.

Divisibility UNO® Game Rules

SETUP:
4-player example



Playing the Game

Each player will attempt to apply the divisibility rules to construct the LARGEST dividend, using only the cards in their hand, that is divisible by the designated **divisor**. It is possible to construct a dividend that uses only a subset of the cards in their hand. *Obviously, the more cards players use and where they are placed in the dividend will improve player scores*.

If they cannot construct ANY dividend using the cards in their hand, they must "Pass." An example of this could be: divisor is 5 or 10, but no 5 or 0 cards are in their hand.

When everyone has had a chance to play (and update their score sheet), the **"used" cards are collected in a discard pile and replaced with new cards from the Uno player deck**. PLAYERS MUST KEEP ANY CARDS in their hand that were not used in the round. ** I.e., at no time should a round begin with players having more than or less than the agreed number of cards! **

A new divisor from the top of the Divisor Pile is selected and turned FACE UP in the center of the table for all to see to start the next round. (It is suggested that students take turns performing this action for each round of play.)

Divisibility UNO® Game Rules

Scoring the Game

The target score should be determined at the start of the game and is dependent on how many cards are going to be used by players in each round.

The formula for determining the target score is: **10^e where *e* = number of cards + 1**

Example of 4-digit dividend game: *e* = 4 + 1 Target score is 10^5 (or 100,000)

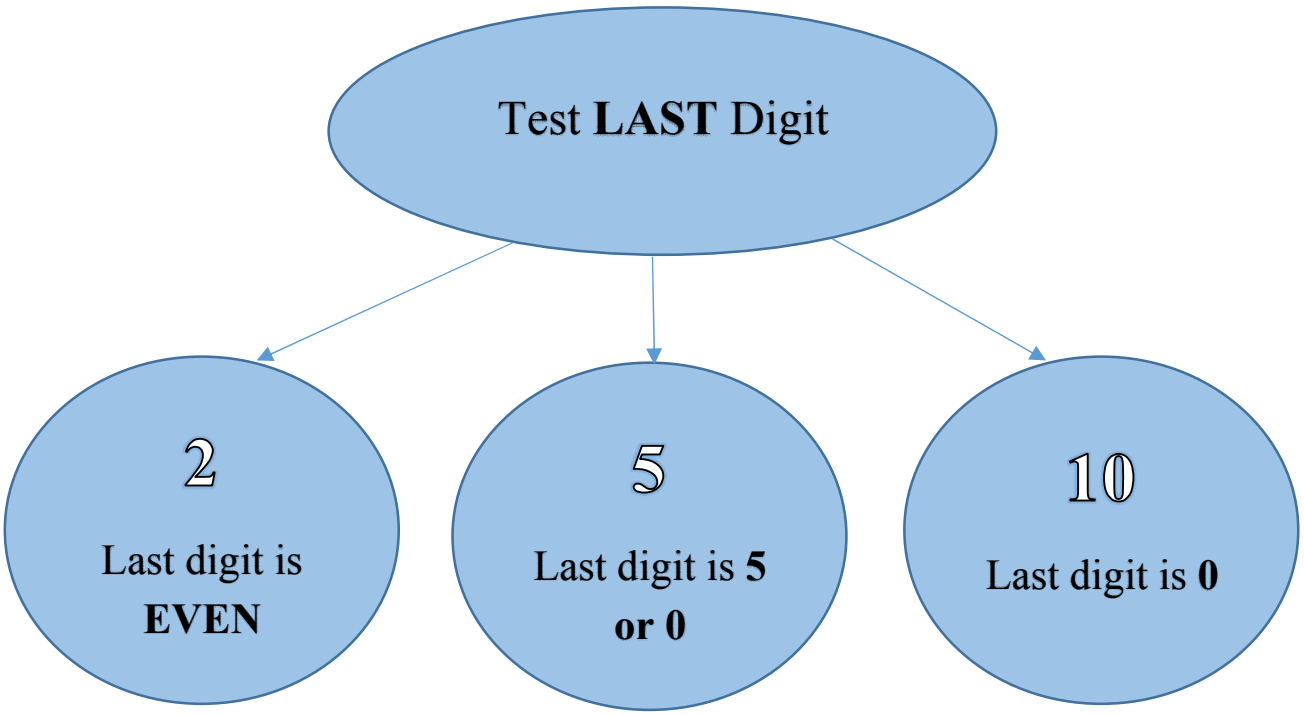
Each round, students will add the dividend they created to their score and KEEP A RUNNING TOTAL. It is not advisable to wait until the end of play to tally the score (note: time is a factor and how will you know the game is "over"?). *Nothing is added or subtracted from their score if a player had to "Pass".*

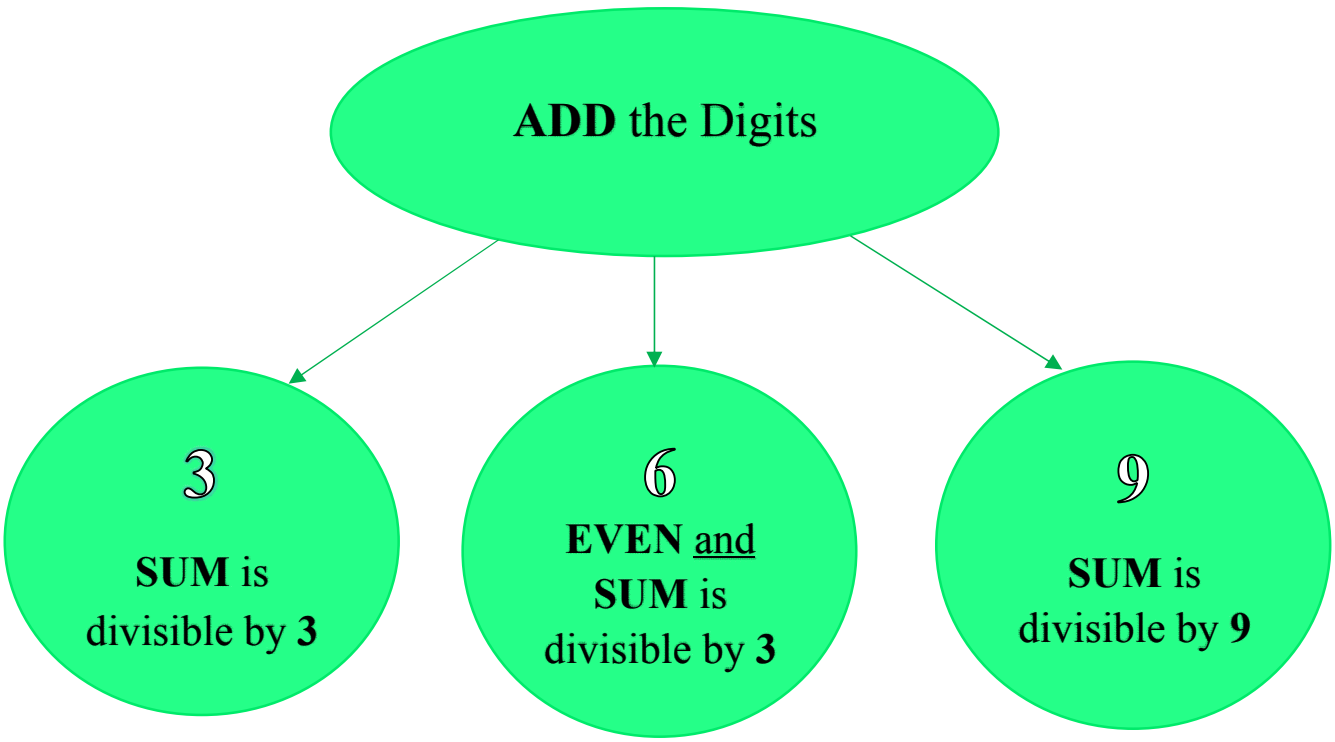
Alternate Scoring Option: An alternative to the above method of scoring is "Race to Zero!" In this adaptation, players all begin with the target score (as described above, e.g., in a 5-digit game, players all begin with one million points) and then SUBTRACT the dividend for each round until they reach zero – or a negative number. *This version of scoring reinforces subtraction with regrouping skills.*

Winning the Game

The first player to reach or exceed the target score is the winner. Players should go around the table announcing their final score, in correct and complete numerical form, to reinforce skills in reading/saying large numbers. (Collect the score sheets if needed for review/assessment.)

Divisibility Tests by Family





Name: _____

Problem-Solving with Divisibility

Date: _____

1. Joe wants to make a 6-digit PIN number, using the digits, 1, 2, 3, 4, 5, and 6, once each. He wants a way to help remember the number. He decides to make a special 6-digit number so that the first digit is divisible by 1, the first two digits are divisible by 2, the first three digits are divisible by 3, the first four digits are divisible by 4, the first five digits are divisible by 5, and the entire 6-digit PIN number is divisible by 6. Help Joe find a 6-digit PIN number that meets the divisibility conditions.

workspace

Conclusion: I think Joe's PIN number should be

_____	_____	_____	_____	_____	_____
1st digit	2nd digit	3rd digit	4th digit	5th digit	6th digit

2. Given: 6 IS a factor of both 14016 and 49320.

Is 6 a factor of 49320 + 14016 and 49320 - 14016? *Provide PROOF of your conclusion below.*

3. In each of the following numbers without doing actual division, determine whether the first number is divisible by the second number:

(a) 3,409,122 6

(b) 17,218 6

(c) 11,309,634 8

(d) 515,712 8

(e) 3,501,804 4

Name: _____KEY_____

Problem-Solving with Divisibility

Date: _____

1. Joe wants to make a 6-digit PIN number, using the digits, 1, 2, 3, 4, 5, and 6, once each. He wants a way to help remember the number. He decides to make a special 6-digit number so that the first digit is divisible by 1, the first two digits are divisible by 2, the first three digits are divisible by 3, the first four digits are divisible by 4, the first five digits are divisible by 5, and the entire 6-digit PIN number is divisible by 6. Help Joe find a 6-digit PIN number that meets the divisibility conditions.

2nd digit must be even {2, 4, 6}

4th digit is even & 3rd and 4th digits make a number divisible by 4 {12, 16, 32, 36}

5th digit is a 5 (since 0 is not an option)

6th digit is even, since sum of the digits is 18 {2, 4, 6}

Therefore, if 2nd, 4th, and 6th digits must be even…and all odd digits must be ODD.

Digits 1 thru 3 must add up to a multiple of 3 {123, 321}

Conclusion: I think Joe's PIN number ~~should~~ COULD be **123654 or 321654**

2. Given: 6 IS a factor of both 14016 and 49320.

Is 6 a factor of 49320 + 14016 and 49320 - 14016? *Provide PROOF of your conclusion below.*

Given: 6 is a factor of both numbers. **49320 = 6 * 8220** and **14016 = 6 * 2336**

49320 + 14016 = 6(8220 + 2336)	Factor	49320 - 14016 = 6(8220 - 2336)
63336 = 6(10556)	Simplify by combining like terms	35304 = 6(5884)
63336 = 63336 True	Multiply	35304 = 35304 True

3. In each of the following numbers without doing actual division, determine whether the first number is divisible by the second number:

(a) 3,409,122	6	Even and sum = 21; IS divisible by 6
(b) 17,218	6	Even and sum = 19; IS NOT divisible by 6
(c) 11,309,634	8	634 IS NOT divisible by 8; 11,309,634 IS NOT
(d) 515,712	8	712 IS divisible by 8; 515,712 IS divisible
(e) 3,501,804	4	04 IS divisible by 4; 3,501,804 IS divisible

Patterns as Process

When you use patterns to organize, analyze, and solve problems, the reasoning process is made easier – and *more efficient*.

Do you **agree or disagree** with this statement?

Consider: If you did not pay attention to, or were unaware of, patterns in number multiples, how would you solve the following "simple" problem when faced with large numbers? How long would it take you? How could you check the *reasonableness* of your conclusion?

Question: Is 2,037,816 divisible by 24? If not, what will the remainder be?

Traditional process: Algorithmic division methods (e.g., long division, short division, partial quotients, but NOT repeated subtraction) will likely be used in Grade 6, with varying degrees of accuracy and speed, dependent upon the experience of the problem solver. These methods also rely, to a large degree, on student skill with multiplication facts and estimation strategies. The algorithmic methods focus attention on the quotient, rather than answering the question that was actually asked…

Checking for reasonableness could involve using a different algorithmic method, multiplying the quotient by the divisor, or employing a calculator.

Process using patterns: Recognizing the divisor, 24, as the product of 3 and 8 is a quick first step. Then solve a related problem: "Can we use the divisibility patterns of 3 and 8 to determine whether the large number is divisible by 24?" (Note: These factors of 24 are *relatively prime*, having no common prime factors, so using these two patterns will be a sufficient test for division by 24.)

Test for 3 – if the sum of the digits is divisible by 3 — Sum is 27; **IS divisible by 3**

Test for 8 – if last three digits create number divisible by 8 — 816÷8=102; **IS divisible by 8**

Since 2,037,816 is divisible by BOTH 3 and 8, it must be divisible by 24 (i.e., 3*8) and the remainder is zero!

In each step of this approach, the process involved fairly simple single-digit division reasoning; mental math alone is often effective here.

The answer could be verified in a simplified two-step division process (using short division), and the quotient identified, as shown below.

```
        6   7   9   2   7   2   r 0          8   4   9   0   9   r 0
3 ) 2   0  23  27   8  21   6        8 ) 6   7  39  72   7   2
```

The two-step process modeled above frequently proves to be more efficient: fewer mistakes, less elapsed time. *Would you be willing to put it to the test?*

Patterns as Process

2,037,816 ÷ 24 = 84,909 no matter which methodology you use!

Summary: The standard algorithm for division is but ONE method for solving problems of this type. There MAY be more flexible and efficient ways available if you exploit divisibility patterns.

Another benefit of using the divisibility patterns is that the remainder is often revealed by the test(s).

Ex 1. Given: Is 1,234,567,890 divisible by 8?

1,234,567,890 ÷ (8) Test for 8: 890 IS NOT divisible by 8 (r: 2)

Therefore, the number is NOT divisible by 8 and the remainder will be 2 (i.e., 2/8 or .25).

Ex 2. Given: Is 1,234,567,890 divisible by 33?

1,234,567,890 ÷ (3*11) Test for 3: sum = 45; IS divisible by 3 (r: 0)

Test for 11: 20 – 25 = -5; IS NOT divisible by 11

(r: 11 – 5 = 6)

Therefore, the number is NOT divisible by 33 and the remainder will be 6 (i.e., 6/11 or $.\overline{54}$).

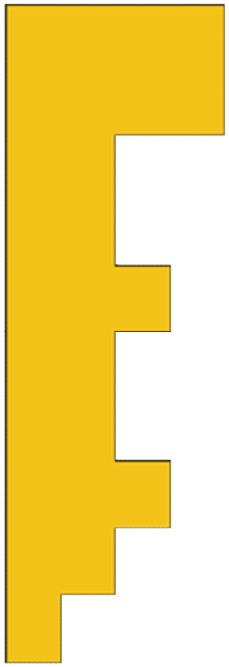
Ex 3. Given: Is 1,234,567,890 divisible by 36?

1,234,567,890 ÷ (4*9) Test for 4: 90 IS NOT divisible by 4 (r: 2)

Test for 9: sum = 45; IS divisible by 9 (r: 0)

Therefore, the number is NOT divisible by 36 and the remainder will be 2 (i.e., 2/4 or .5).

The SHAPE of Primes to 100



2	3	5	7
11	13	17	19
23	29		
31	37		
41	43	47	
53	59		
61	67		
71	73	79	
83	89		
97			

The SHAPE of Primes to 100