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THE 2022 ROSENTHAL PRIZE
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Billiard Ball Problem

Dan Finkel

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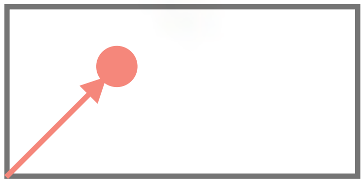
Task at a glance

The Billiard Ball problem starts by imagining an ideal sort of pool table, and is motivated with a simple intent: how do we understand the mechanics of this idealized world of ricochets?

The ball always launches from the bottom left corner of a rectangular table at 45 degrees, and bounces until it reaches a corner. Surely there are patterns at play that will help us make meaningful predictions. Finding them will take some perseverance, and require us to get smart about organizing our search for structure.



Big questions:

Day 1. When will billiard ball paths describe the same shape, despite the size of the billiard table being different?

Day 2. How can we predict what corner the billiard ball will end up in?

Day 3. How can we put it all together to prove our conjectures are true?

Lesson Goals

This is a problem designed to reward and encourage organization, pattern-seeking, conjecturing, and breaking a complex situation down into smaller cases. As soon as students can draw straight lines at 45 degrees, they have access to the puzzle, but cracking it is much trickier!

The pacing of the problem is designed to get students engaged in meaty problem solving. While there are substantial discoveries relating to ratio and proportion involved in solving this problem, the real power comes from students using a new organizational structure to look for patterns, and building perseverance as partial explanation emerge.

Common Core Standards for Mathematical Practice

CCSS.MATH.PRACTICE.MP1
Make sense of problems and persevere in solving them.

Even though the rules of the puzzle are straightforward, the behavior of the billiard ball on different tables is hard to predict. Students must break a large problem into smaller pieces, make sense of the patterns they find, conjecture when the patterns continue, and revise their conjectures when counterexamples emerge.

CCSS.MATH.PRACTICE.MP2
Reason abstractly and quantitatively.

Some billiard tables have the same designs created by the path of the billiard ball. When will tables "match" in this way? Students must flip from examining specific cases to working out general principles to explain the behavior.

CCSS.MATH.PRACTICE.MP3
Construct viable arguments and critique the reasoning of others.

Students will move from finding patterns to articulating conjectures and defending them. The best critique is a counterexample, and many will emerge as students will find evidence that topples promising conjectures and breaks apparent patterns.

CCSS.MATH.PRACTICE.MP6
Use appropriate tools strategically.

All students will learn the value of a 2-dimensional table in this lesson. They'll also see that they can be strategic in which cases they decide to tackle next. Filling out early rows of the table vs. a scattershot approach can lead to very different pathways through the universe defined by this problem.

CCSS.MATH.PRACTICE.MP6 Attend to precision.
Students will inevitably need to check that they track the path of the billiard balls precisely. Some counterexamples may prove to be due to faulty drawing.

CCSS.MATH.PRACTICE.MP7 Look for and make sense of structure.
This lesson is, at heart, a deep dive into the patterns and structure at play in the billiard ball problem. Figuring out the structures that determine the paths of the billiard ball—due to the ratios of the table lengths or the reflective structure of the ball's path—is the key to understanding the heart of the problem.

Common Core Standards for Mathematical Content

CCSS.Math.Content.6.RP.A.1
Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities.

CCSS.Math.Content.7.RP.A.2
Recognize and represent proportional relationships between quantities.

- CCSS.Math.Content.7.RP.A.2.a
  Decide whether two quantities are in a proportional relationship

CCSS.Math.Content.7.G.A.1
Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

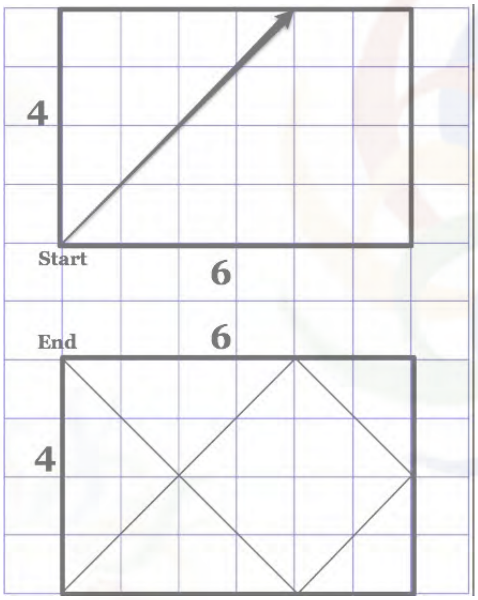
Day 1 - Paths

Setup

Make graph paper available to all students.

Use large graph paper at the board to demonstrate the mechanics of the problem. Projecting a grid onto a white board is an excellent option. You can draw an imitation of graph paper if you're using a white board or blackboard—just make sure your drawing is reasonably neat.

The Launch

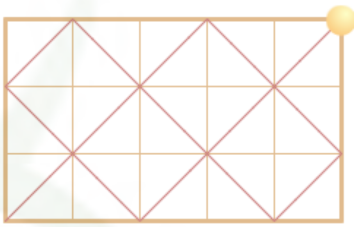


Ask students to volunteer two numbers between 3 and 8 from the class, and draw a rectangle with those dimensions. In our example at the right, students gave the numbers 4 and 6. This rectangle will be the "billiard table" or "pool table" for our example.

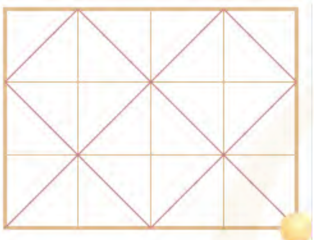
A "ball," modeled by a straight line, is launched at 45 degrees from the bottom left corner, which means it cuts through every square of your graph paper exactly along the diagonal. You can ask the class: what happens when it hits the side of the table? The answer is that it bounces off at the same angle and continues along the diagonal of the square grid.

Do a few more tables with students, encouraging them to draw the billiard ball paths on their own graph paper as you work at the front of class. The work of correctly drawing the billiard ball paths is critical to exploring this problem, and it's worth taking the time to make sure everyone can do it consistently! Let students name some of the shapes they get - this will often happen unprompted, occasionally with help from a little decorative shading. For example:

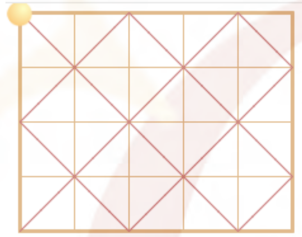
The Fish
2 by 3



Dominoes
3 by 5



The Heart
3 by 4

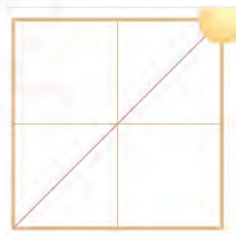


The Pretty Fish
4 by 5

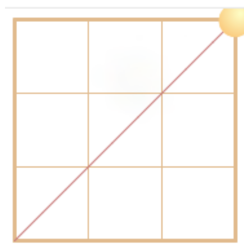
As the mechanism of billiard ball travel becomes clear to students, note that it might be possible for certain tables to give the same shape path despite having different dimensions. For example, the "straight shot" path that doesn't ricochet at all seems like it could occur on more than one table, doesn't it? Challenge students to work with a partner or table group to find other tables with the same shape path.

After a minute working on their own, students should have "rule" by which they can describe other "Straight Shot" tables, namely: any table with the same height and width will be in the "Straight Shot" family. Specific tables that look like the Straight Shot include the 1 by 1, 2 by 2, 3 by 3, and so on.

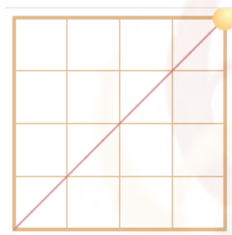
Straight Shot
1 by 1



Straight Shot
2 by 2



Straight Shot
3 by 3

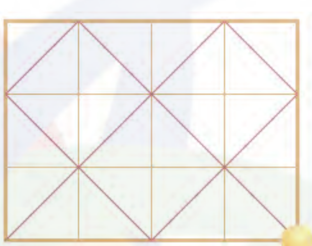


Straight Shot
4 by 4

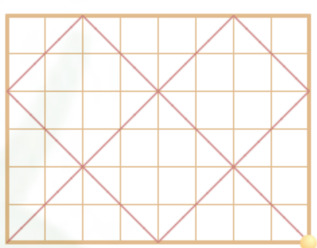
Now pick another shape, say, the "Heart." It's defined by a 3 by 4 table. But maybe there are other tables that give the same shape as well? Challenge students to find at least three other tables that make the same "Heart" shape.

The Work

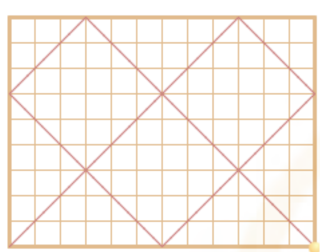
Transition to students creating their own billiard tables and searching for their own lists of tables that match a given shape. Challenge them to create at least three new tables of each type you look at. If they think they have a way to do it, encourage them to double check by actually working out the billiard ball paths on the tables!



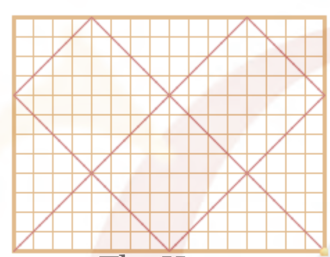
The Heart
3 by 4



The Heart
6 by 8



The Heart
9 by 12



The Heart
12 by 16

Check in with students to confirm they are drawing the shapes correctly, and help them articulate larger conjectures about what's actually happening. Ideally, most will be able to generate more tables within a family as class draws to a close, even if they're still grasping for how to put it into words.

Prompts and questions

- Are you sure you drew that path correctly? Check that again.
- Great result! Make sure you add that to your list.
- Which table do you want to try next?
- What's the relationship between those two tables?
- Why do you think those two gave you the same shape?
- How did you guess to try a 24 by 32 table? How did you know it would give you a fish?

The Wrap

Give one last challenge to students—"Dominoes," say, which results from a 3 by 5 table—and ask them to find three tables with the same shape, and also to describe how the tables are related.

Ideally, students will be able to generate the dimensions that give the matching table design, and articulate what's happening. Students may say things like:

- You can double both dimensions
- You can multiple both dimensions by anything
- You can count by 3s and count by 5s
- The difference between the dimensions goes up by 2, 4, 6, etc.

Tables that give "Dominoes"
3 by 5
6 by 10
9 by 15
12 by 20
15 by 25
…

Clearly articulating the idea of similarity between tables and billiard ball paths may be tricky for the students. You can end class by letting them mull over the various versions of what they've determined at this point, and plan to formalize the idea of similarity on Day 2.

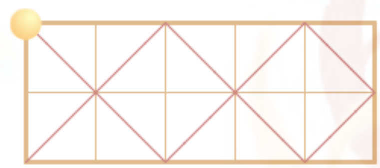
Day 2 - Corners

Setup

In addition to graph paper, have the billiard ball data collection chart ready to hand out to students.

The Launch

Take an example from Day 1 and introduce the idea of similarity. Two tables are similar if their dimensions are in the same ratio, or equivalently, if one is a “scaled up” or “zoomed in” version of the other. Similar tables also seem to give us similar billiard ball paths, or paths in the same “family” of shape designs that we saw on Day 1.

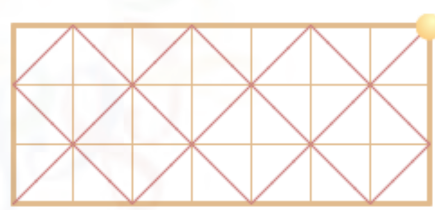


The Glasses
2 by 5

As a warmup, ask students to find three tables that are similar to “The Glasses,” given by a 2 by 5 table. Students will hopefully recall relatively quickly that they can build similar tables using the strategies they discovered on Day 1, and offer a list of tables including 4 by 10, 6 by 15, 8 by 20, and so on.

Now pivot to the big question for today. When you’re playing pool or billiards, you need to be able to call your shot. In our context, that means you can shoot the ball and call what corner it will hit. For a 2 by 5 table, we see that the ball will end up in the top left (TL) corner. But what about a 3 by 7 table? Take predictions for what “pocket” (i.e., corner) the ball will end in.

Once predictions have been taken, have students map out the path of a ball on a 3 by 7 table on their own while you do it at the board. Everyone should agree that it ends in the top right corner.



On a 3 by 7 table, the ball ends in the top right

Now we have a new question, and it will drive the entire exploration today:

Given any billiard ball table, how can you predict what corner the billiard ball will end up in?

Before you release students to work on their own, show them the data collection sheet. Tell them that the difference between solving tough problems like this and getting stuck can come down to keeping organized and attacking them systematically.

3							TR
2							
1							
	1	2	3	4	5	6	7

We enter “TR” for “Top Right” on the data collection sheet in the cell for a 3 by 7 billiard table

It will be helpful to fill out more of the cells of the data collection sheet together with students, using the data you've gathered from Day 1. Encourage them to tell you what goes where, and to fill the data in on their own sheets too. Putting all that data in the table, you may have something that looks like this.

10										TR
9									TR	
8								TR		TL (pretty fish)
7							TR			
6						TR		BR (heart)	TL (fish)	
5					TR					
4				TR	TL (pretty fish)	TL (fish)				TL (glasses)
3			TR	BR (heart)			TR			
2		TR	TL (fish)		TL (glasses)					
1	TR									
	1	2	3	4	5	6	7	8	9	10

Now release students to work on their own, with a goal of filling out the rest of the table, and better,

The Work

By this point, students should be proficient at drawing billiard ball paths, and also understand how the data collection sheet works. They can work on their own, in pairs or trios. As students work, move between the groups to observe what students are trying and give hints or guidance as necessary. Students may take many different approaches to attacking the problem, and a number of them might prove useful.

For students who are having trouble, you can nudge them toward one of these two approaches.

Approach #1.
Start simple, and build from there. Large tables may be tricky, so what's the simplest place we could begin? A 1 by 1 table. From there, we can try a 1 by 2 table, a 1 by 3 table, and continue until the row is complete. What patterns can we see, and are there reasons that pattern should exist? Now we can continue to row 2, and try a 2 by 1 table, a 2 by 2 table, a 2 by 3 table, and so on. This approach should create a lot of momentum!

Approach #2.
Students may remember from Day 1 that certain tables ones show up more than once. For example, the "fish" shape occurs in the 2 by 3 as well as the 4 by 6 table. The "heart" occurs in the 3 by 4 as well as the 6 by 8. Challenge students to classify more cases where they would see the fish shape, or the heart shape. Combining from Approach #1, how do we extend the information about row 1 to tell us something about row 2, or other rows? The cells with the same colors below represent similar table sizes - we actually get half of row 2 for free once row 1 is done!

2		TR		TL		TR		TL		TR
1	TR	TL	TR	TL	TR	TL	TR	TL	TR	TL
	1	2	3	4	5	6	7	8	9	10

Other approaches may prove useful as well, such as considering how a 3 by 2 table relates to a 2 by 3 table. We'll consider some other less-commonly discovered approaches on Day 3.

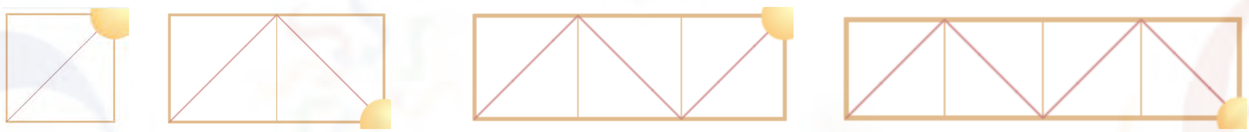
Prompts and questions

- Are you sure you drew that path correctly? Check that again.
- Great result! Did you enter that data in your table already?
- Which table do you want to try next?
- What corner do you think the ball might end up in on that table?
- What about a 6 by 5 table?
- Let's start simpler… what's an easier table we could look at?
- What about a 1 by 1 table?

- What about a 1 by 2 table? A 1 by 3 table? I bet you could fill out this entire row. Let me know if you find any patterns.
- Good conjecture! Do you think it will hold for the next row of your table?
- You're convinced you're right, huh? Try to fill out row 4, and tell me if your conjectures still hold.

The Wrap

Ask students to help you fill in the rest of the table. Do row 1 first, and then ask what predictions they have about how the pattern would continue. In this case, there's an ABAB pattern that seems consistent. Can students defend why it holds?



Repeat for row 2. The scaling argument can help us fill in half of the row (see Approach #2 above). How does the other half get filled in? Can students defend why this is so?

Continue as time permits, filling in rows and soliciting observations about patterns in the data collection sheets from students. Do any of these observations extend to conjectures or predictions? If so, encourage students to break them with counterexamples, or test them with subsequent rows of the table.

End by having each students write their favorite conjecture down for how to predict what corner a billiard ball will end up in. You could even ask them to test their conjecture on larger tables, like 3 by 17 (Top Right), 6 by 11 (Top Left) and 4 by 22 (Top Left).

Ideally students will have arrived at (or close to) the following conjectures, which, it turns out, are true.

Conjecture 1. Two tables give the same shape if their sides are proportional. In particular, this means a billiard ball end in the same corner for proportional tables.

Example: A 5 by 8 and a 15 by 24 table both have the billiard ball end up in the same corner.

Conjecture 2. The oddness or evenness of a billiard tables dimensions is the key to determining what corner a billiard ball ends in.

- Odd by Even table - ball ends in Bottom Right (BR)
- Even by Odd table - ball ends in Top Left (TL)
- Odd by Odd table - ball ends in Top Right

In the Even by Even case, look at a proportional table with smaller sides and apply one of the top three rules.

Example: To find what corner a ball on a 22 by 36 table ends in, first note that it will be the same as the proportional 11 by 18 table. This is Odd by Even, so we predict that the ball will end in the Bottom Right corner.

We'll work tomorrow on arguments for why these conjectures hold.

Tips for the Classroom

1. Some students have a surprising amount of trouble drawing the path of the billiard ball. Make sure all your students can actually, correctly derive the path. Graph paper helps!

2. Certain conjectures are easy to come by, but can also be instructive. The 1 by 1, the 2 by 2, the 3 by 3 table, etc., are all similar shapes, and clearly the billiard ball ends up in the top right corner. Use these observations from students to ask them what's like the 1 by 2, or the 2 by 3, etc. table.

3. The "4" row of the table provides lots of counterexamples. You can encourage students to try that one if they're overly convinced in their conjectures.

4. There's a deep idea, related to attacking this problem, of imagining the billiard ball of going into an identical table "through the looking glass" of the surface it reflects off of. This idea can provide a completely different, very powerful approach, if you unpack it. See Day 3.

Name________________________________

Billiard Ball Data Collection

What corner does the ball end in?

Height

10										
9										
8										
7										
6										
5										
4										
3										
2										
1										
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**

Width

Top Left = TL Top Right = TR

Bottom Left = BL Bottom Right = BR

Name________________________________

Billiard Ball Data Collection - Teacher Key

What corner does the ball end in?

Height										
10	TL	TR	TL	BR	TL	TR	TL	BR	TL	TR
9	TR	BR	TR	BR	TR	BR	TR	BR	TR	BR
8	TL	TL	TL	TL	TL	TL	TL	TR	TL	TL
7	TR	BR	TR	BR	TR	BR	TR	BR	TR	BR
6	TL	TR	TL	BR	TL	TR	TL	BR	TL	TR
5	TR	BR	TR	BR	TR	BR	TR	BR	TR	BR
4	TL	TL	TL	TR	TL	TL	TL	BR	TL	TL
3	TR	BR	TR	BR	TR	BR	TR	BR	TR	BR
2	TL	TR	TL	BR	TL	TR	TL	BR	TL	TR
1	TR	BR	TR	BR	TR	BR	TR	BR	TR	BR
	1	2	3	4	5	6	7	8	9	10

Width

Top Left = TL　　Top Right = TR

Bottom Left = BL　　Bottom Right = BR

Name________________________________

Billiard Ball Data Collection - Teacher Key

What corner does the ball end in?

Height

10	TL	TR	TL	BR	TL	TR	TL	BR	TL	TR
9	TR	BR	TR	BR	TR	BR	TR	BR	TR	BR
8	TL	TL	TL	TL	TL	TL	TL	TR	TL	TL
7	TR	BR	TR	BR	TR	BR	TR	BR	TR	BR
6	TL	TR	TL	BR	TL	TR	TL	BR	TL	TR
5	TR	BR	TR	BR	TR	BR	TR	BR	TR	BR
4	TL	TL	TL	TR	TL	TL	TL	BR	TL	TL
3	TR	BR	TR	BR	TR	BR	TR	BR	TR	BR
2	TL	TR	TL	BR	TL	TR	TL	BR	TL	TR
1	TR	BR	TR	BR	TR	BR	TR	BR	TR	BR
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**

Width

Note: this ray of slope 1/2 gives similar tables corresponding to the “fish” path.

We can plot lines of any rational slope through similar tables.

Day 3 - Proofs & Extensions (optional)

Setup

Students should have graph paper, the billiard ball data collection chart, and whatever conjectures they articulated on Day 2.

The Launch

Today we'll help students try to prove their conjectures about Billiard Tables. Depending on your class, you can either share the following ideas one at a time from the board, or send students to work at different stations with one proof idea handout at the center of each station. Students should turn in a written argument fleshing out a proof idea or working out an extension by the end of class.

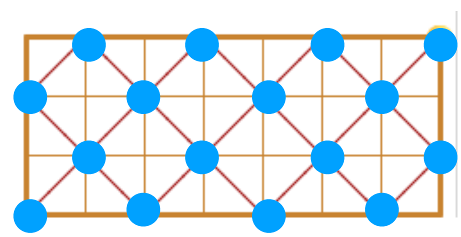
For students who aren't ready to prove the central conjectures from the Wrap on Day 2, they can also work on applying them to predict the corners for more, larger tables.

You can also give students extension problems to work on as an alternative or supplement.

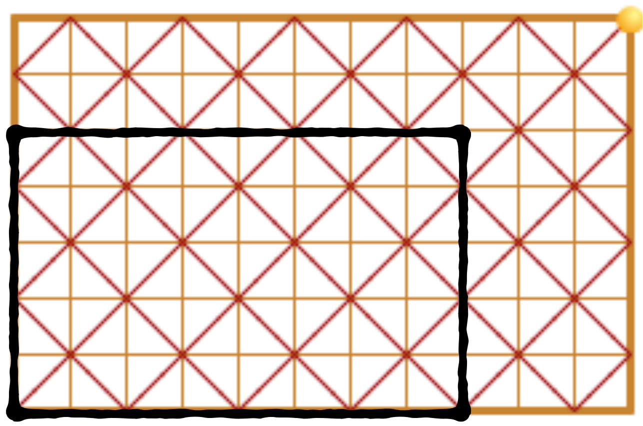
Proof idea 1.
Imagine the billiard ball is a laser beam and the edge of the table is a mirror. Instead of bouncing off, you could instead go "into the mirror" and imagine what's happening in the mirror image. Keep going and you end up with a straight path through reflected tables. Using this framework, can you explain why the patterns in your table hold?

(Note: proof idea 1 is probably the deepest and most challenging. Don't expect all students to have success with this one.)

Proof idea 2.
Try coloring in the intersection points the billiard ball hits on its path. It looks like will create a checkerboard pattern on those points. Does it make sense that it would? Can that help us predict what corner the ball will end up in?



Proof idea 3.
Consider a large table, and image drawing a smaller table in the bottom left hard corner. Here, for example, is a 5 by 8 table inside a 9 by 11 table. Note that the path from the larger table is still accurate on the smaller table. When does this work, and when doesn't it? Can this help us?



Challenge 1.
Given any table, can you predict the number of bounces the ball will make before it hits a corner?

Challenge 2.
Given any table, can you predict the number of times the ball will intersect its own path before it hits a corner?

Challenge 3.
Given any table, can you predict the total distance the ball will travel before it hits a corner?

A good hint for all three extensions: solve for tables with height 1 first, then height 2, and then continue from there.

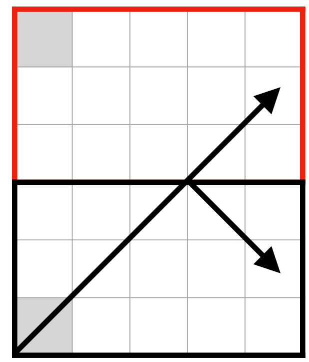
The Work
Students can work on fleshing out a proof or solving an extension. They can work alone or in a small group, but everyone should plan to turn some of their own work in by the end. The teacher can devote the class to helping students tackle the extensions or write up proofs.

If full arguments prove too difficult for students, ask students to write down a solution for how to use the patterns and conjectures to make a prediction for what corner a billiard ball would end up in on a 36 by 98 table. This is too big to do by hand, so they'll have to apply the conjectures from Day 2.

Proof idea 1 Handout

Imagine the billiard ball is a laser beam and the edge of the table is a mirror. Instead of bouncing off, you could instead go “into the mirror” and imagine what’s happening in the mirror image. Keep going and you end up with a straight path through reflected tables. Using this framework, can you explain why the patterns in your table hold?

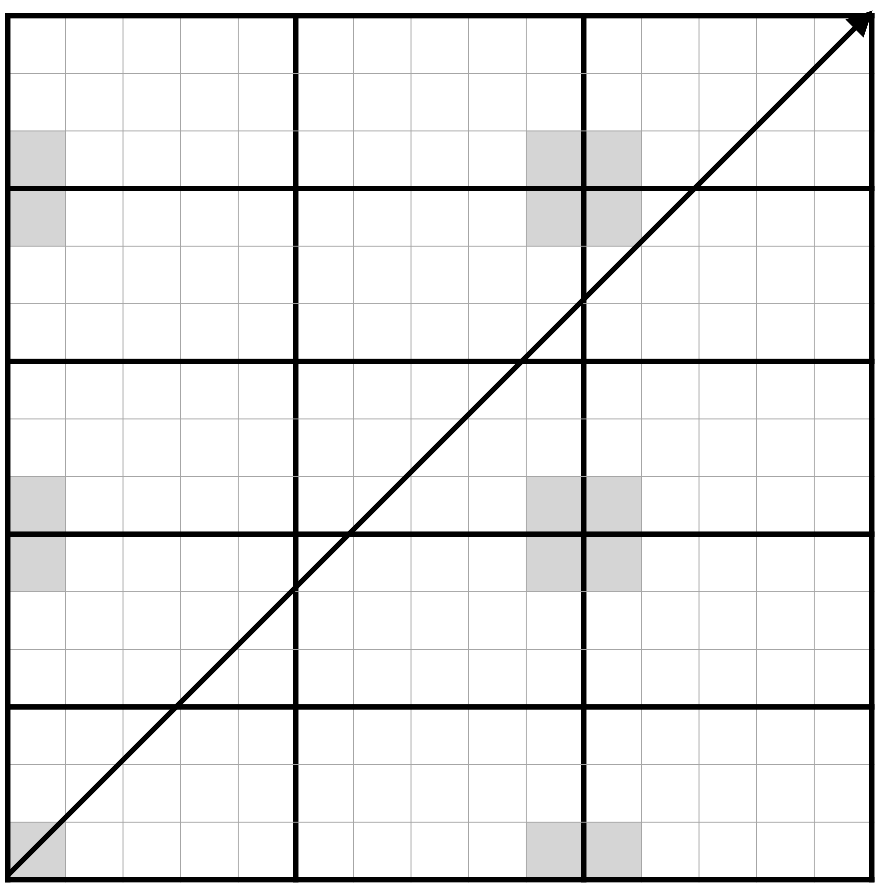
Consider this 3 by 5 table, for example. Instead of the ball bouncing off the edge, image it went through the wall into a mirror image of the table. (We shaded in a square to track how the table reflects).



Tile enough of the reflections of the tables together, and we can track the entire path of the billiard ball as a straight line through table reflections. The billiard ball will cut across 15 squares to reach the corner.

Try this for a 2 by 3 table. Do you see how this idea can help predict the final corner?

Expand this argument to prove a conjecture about what corner the billiard ball will end in.



Proof idea 2 Handout

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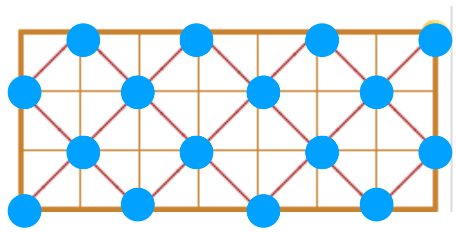
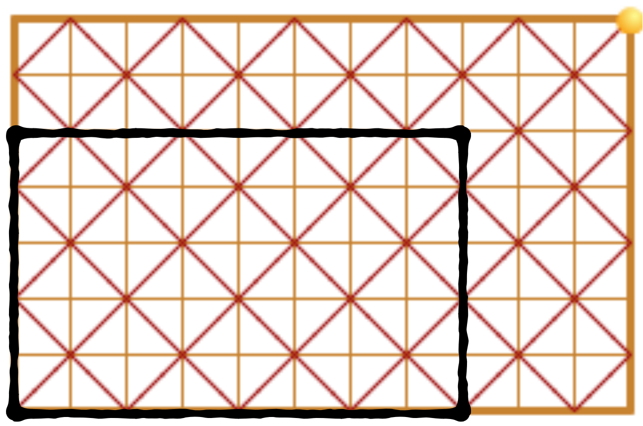


Expand this argument to prove a conjecture about what corner the billiard ball will end in.

Proof idea 3 Handout

Consider a large table, and image drawing a smaller table in the bottom left hard corner. Here, for example, is a 5 by 8 table inside a 9 by 11 table. Note that the path from the larger table is still accurate on the smaller table. When does this work, and when doesn't it? Can this help us?



Expand this argument to prove a conjecture about what corner the billiard ball will end in.

Billiard Ball Challenges Handout

Answer any of these questions for small tables to try to find a general pattern.

1. Given any table, predict the number of bounces the ball will make before it hits a corner.

2. Given any table, predict the number of times the ball will intersect its own path before it hits a corner.

3. Given any table, predict the total distance the ball will travel before it hits a corner.

Hint: solve for tables with height 1 first, then height 2, and then continue from there.

The Wrap

Students turn in their work for today. Ask students how this experience compared with math they've done in the past. Listen to their comments and discussions. Close by emphasizing that this kind of experience—defining a new type of problem to work on, exploring it without knowing where it will go, finding patterns, making conjectures, creating arguments and proofs to understand what's really happening—is exactly what mathematicians do.

In fact, the work of analyzing billiard balls flying around billiard tables not only has surprising applications (like modeling movements of gasses in physics), but leads us into some cutting-edge current mathematics, like the work of Maryam Mirzakhani, the first women to receive the Fields Medal, math's highest honor.

Students who are interested in learning more about Maryan Mirzakhani might start with Jordan Ellenberg's article in Slate: https://slate.com/human-interest/2014/08/maryam-mirzakhani-fields-medal-first-woman-to-win-maths-biggest-prize-works-in-dynamics.html.

Imagining billiard tables may seem like an odd diversion from normal mathematics, but in fact, this is the kind of diversion that takes us to places where new mathematics gets developed.