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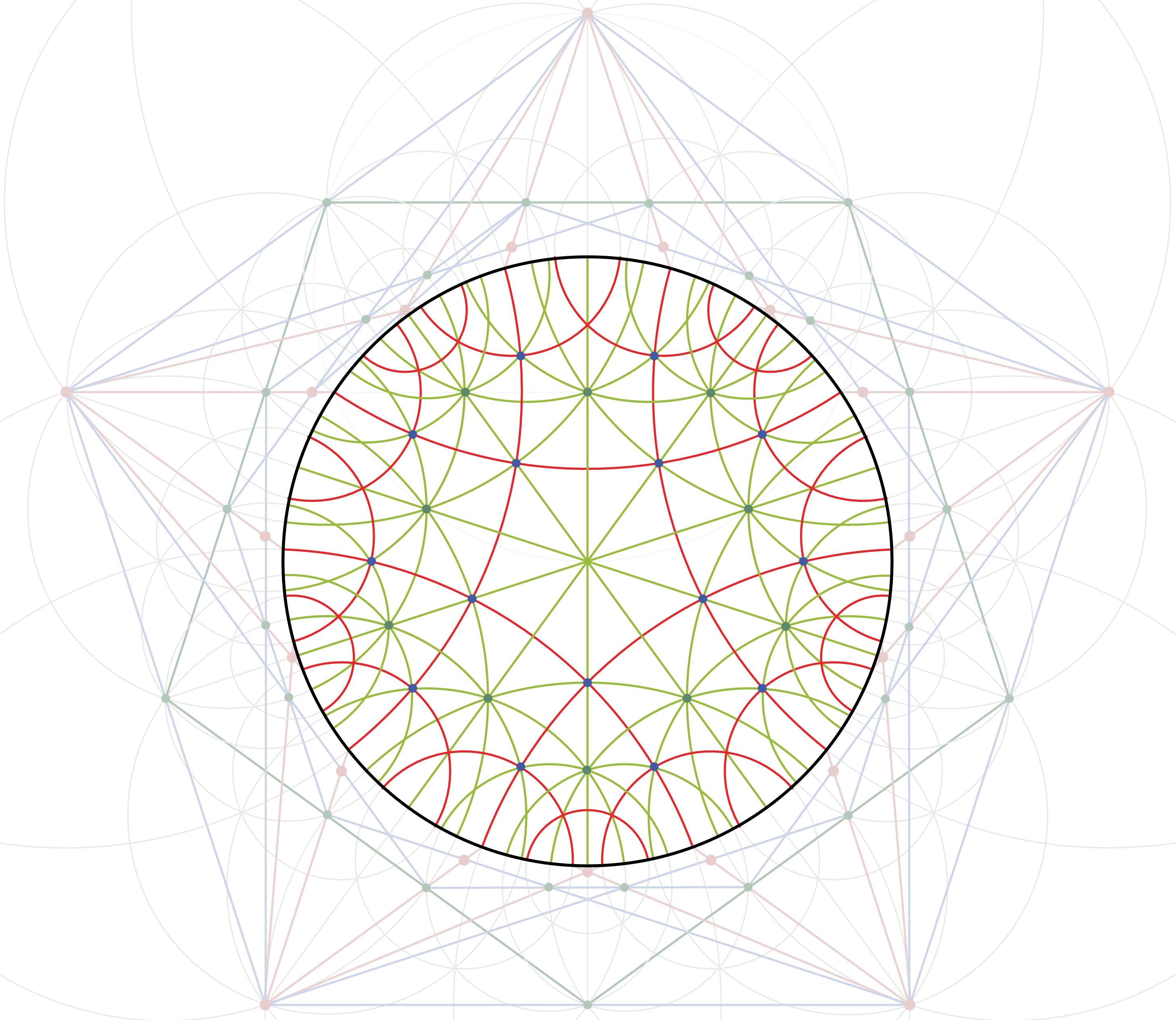
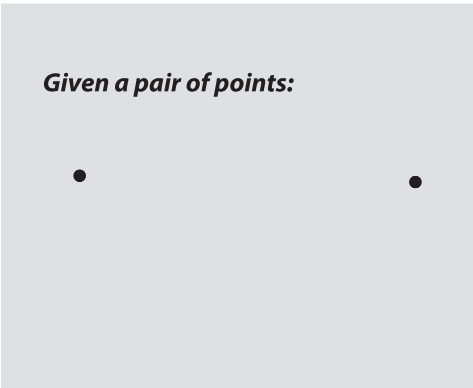
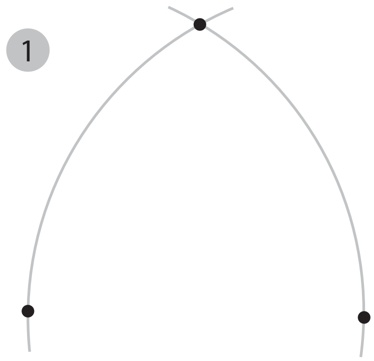


Euclid's Elements, Summarized

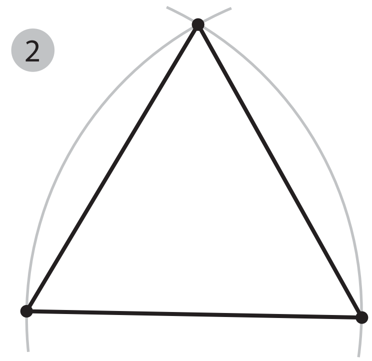
How can you construct a perfect equilateral triangle using your straight edge and compass?

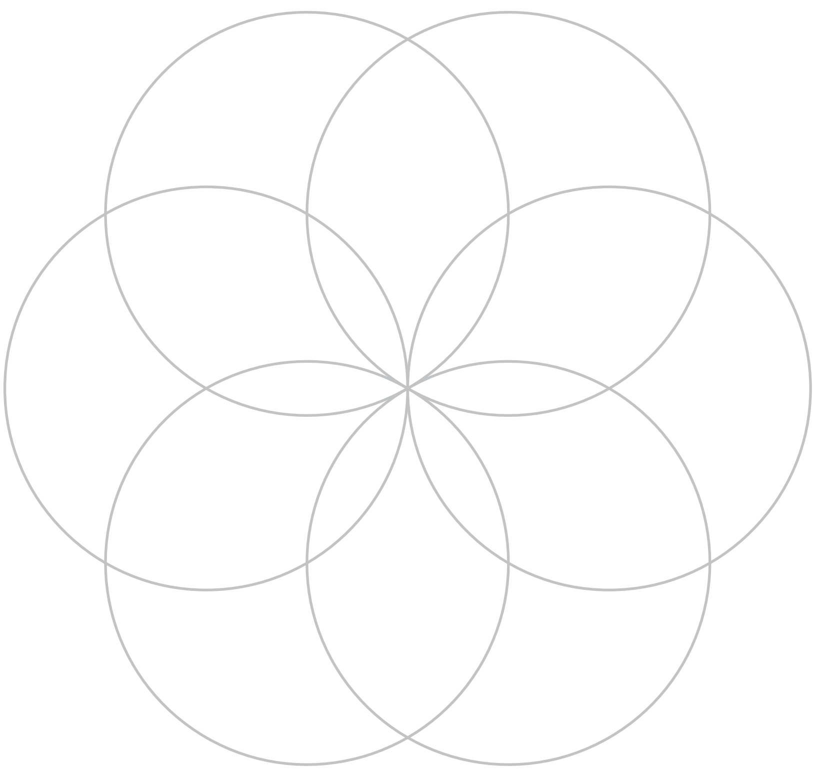




Draw it yourself!

*What's the **proof** this really works?*





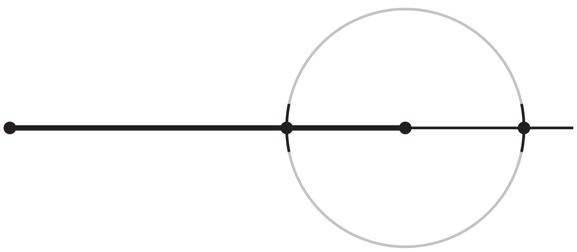
As an extra, can you construct a regular hexagon?

What other neat constructions can you come up with?

Given:

Here are step-by-step instructions for constructing a square — but ***why do we really obtain a polygon with four equal sides and four right angles?*** *What do we need to use to be able to say this?*

1



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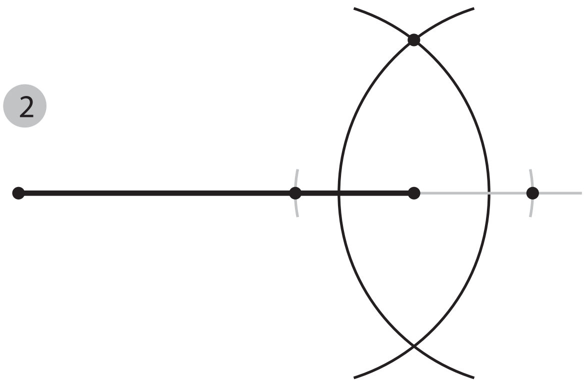

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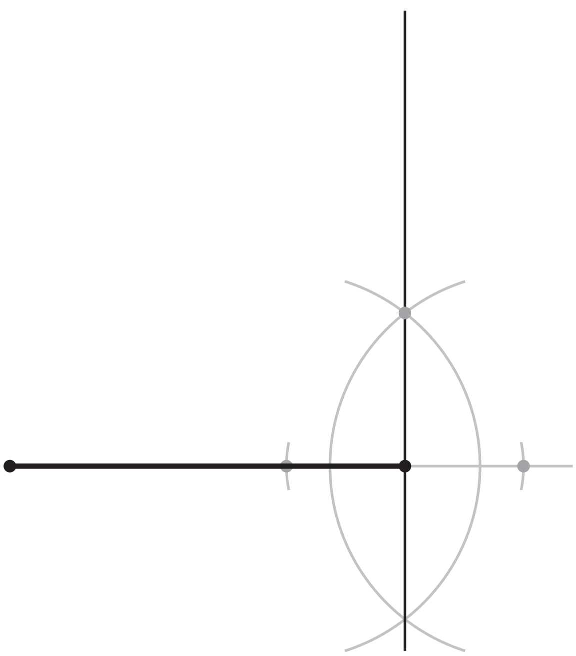

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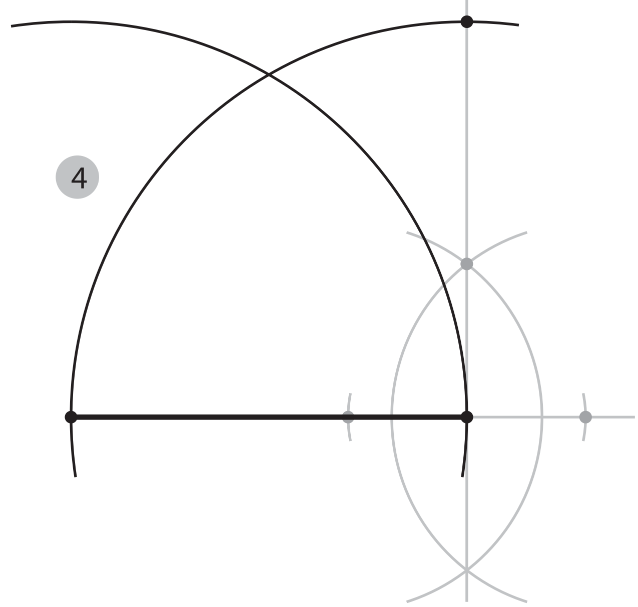

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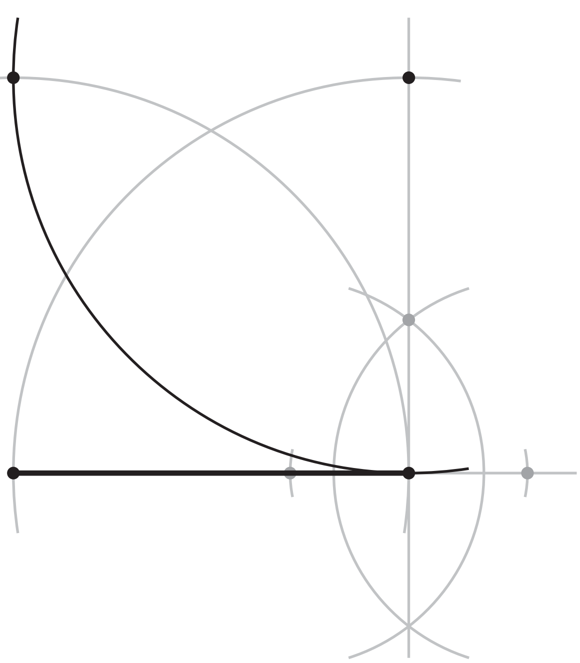

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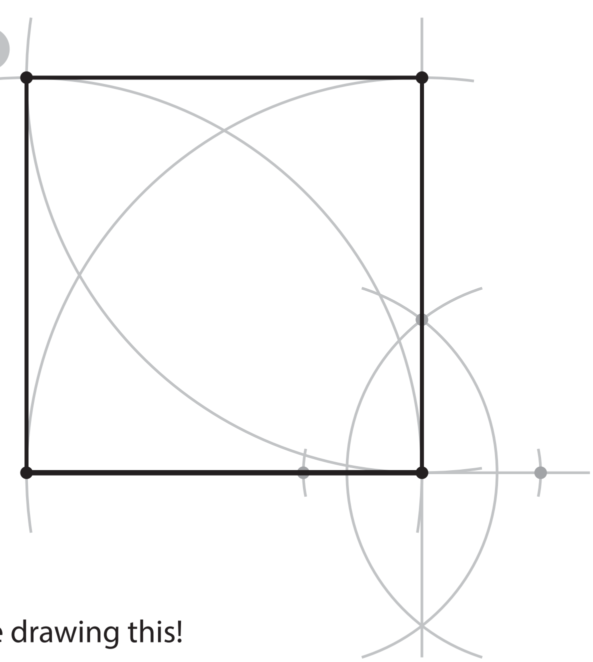
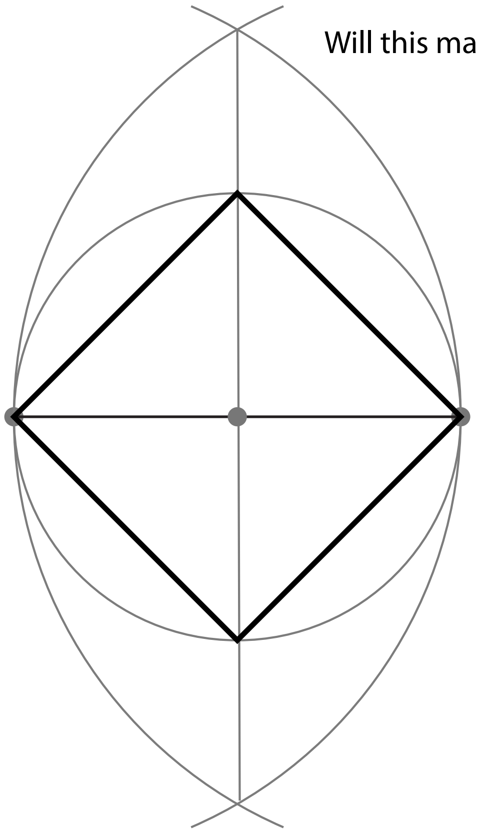


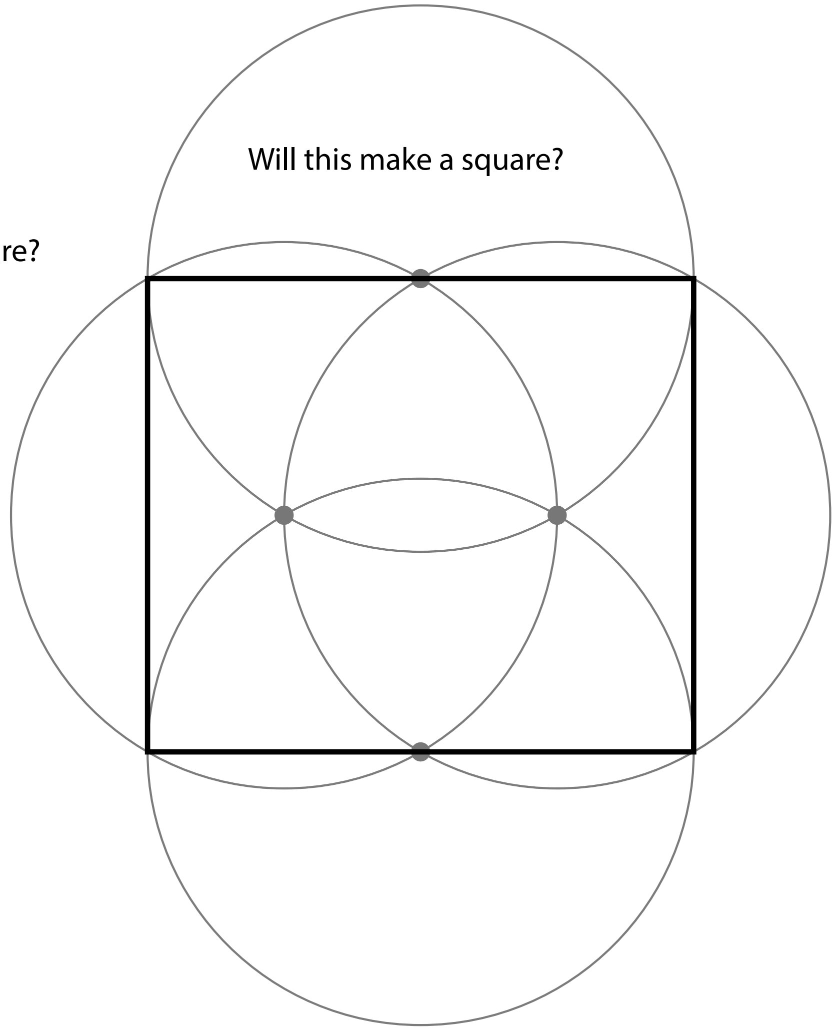
Practice drawing this!

Mysterious Constructions!

Will this make a square?

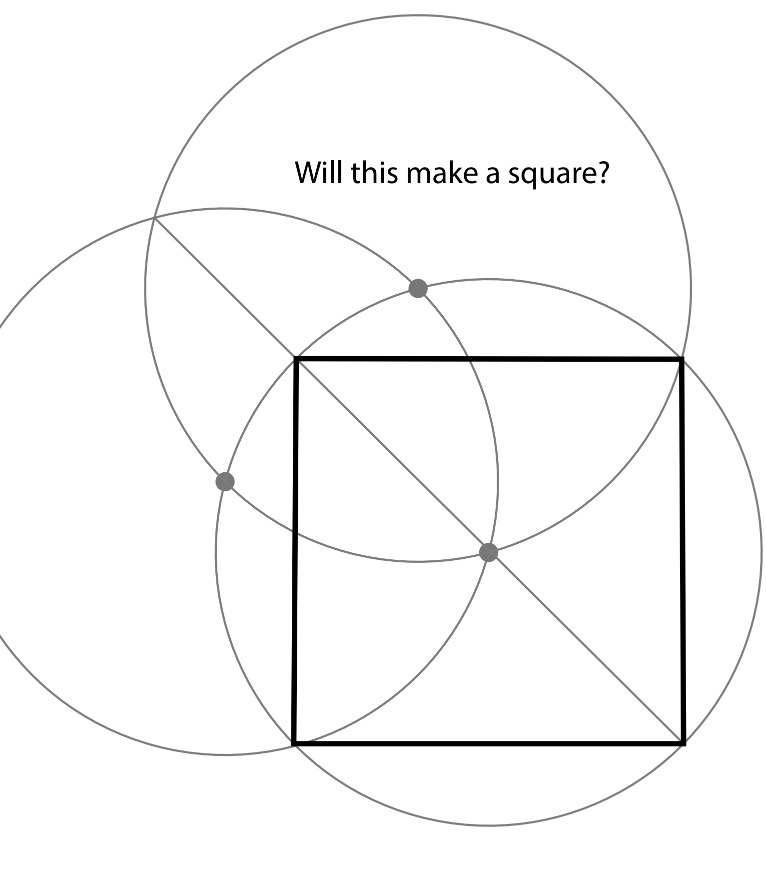


Will this make a square?

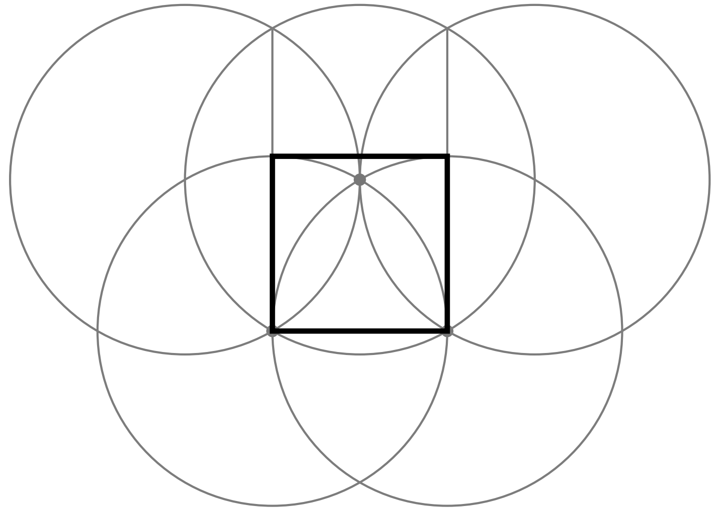


Can you prove it?

Will this make a square?



How about this?



(Olin Ottermann-Strauss, Epsilon Camp 2014)

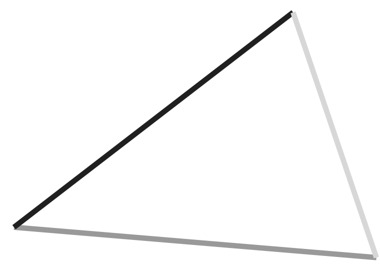
The Triangle Inequality, and
how to construct a triangle given its sides

Euclid 1.20 is the famous Triangle Inequality: If the sides of a triangle have lengths A, B, and C

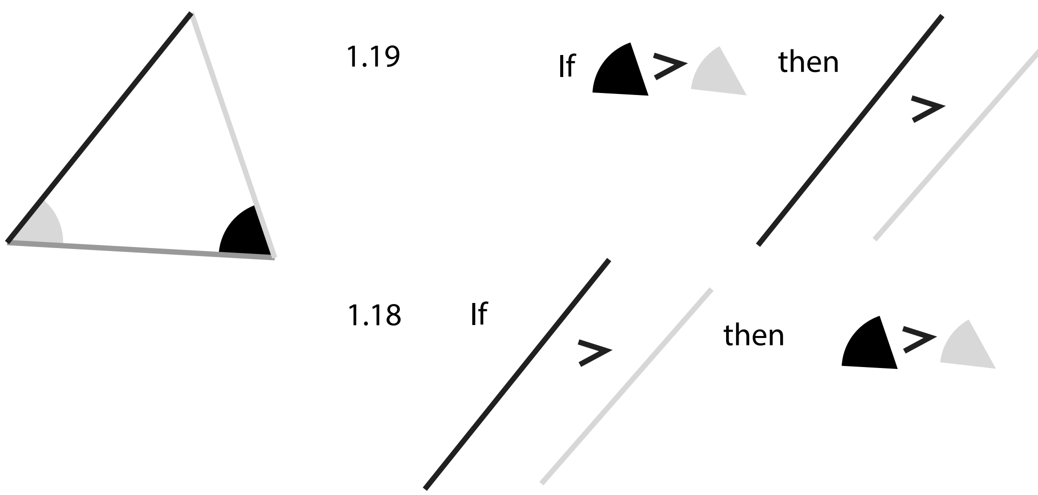
A+B must be less than C,

B+C must be less than A,

C+A must be less than B.



In order to pull this off, Euclid uses 1.19, which uses 1.18, which uses 1.16.



And Euclid 1.22 tells us how to construct a triangle with three specified side lengths (if they satisfy the triangle inequality). For practice, construct a triangle with these three side lengths:

We’ve seen

1.26 ASA, AAS!

SAS (”Side-angle-side” Proposition 1.4)

SSS (”Side-side-side” Propositions 1.7 & 1.8)

Running through the remaining possibilities, why is AAA just false? Can you find a pair of triangles with all angles the same, but with differently sized edges?

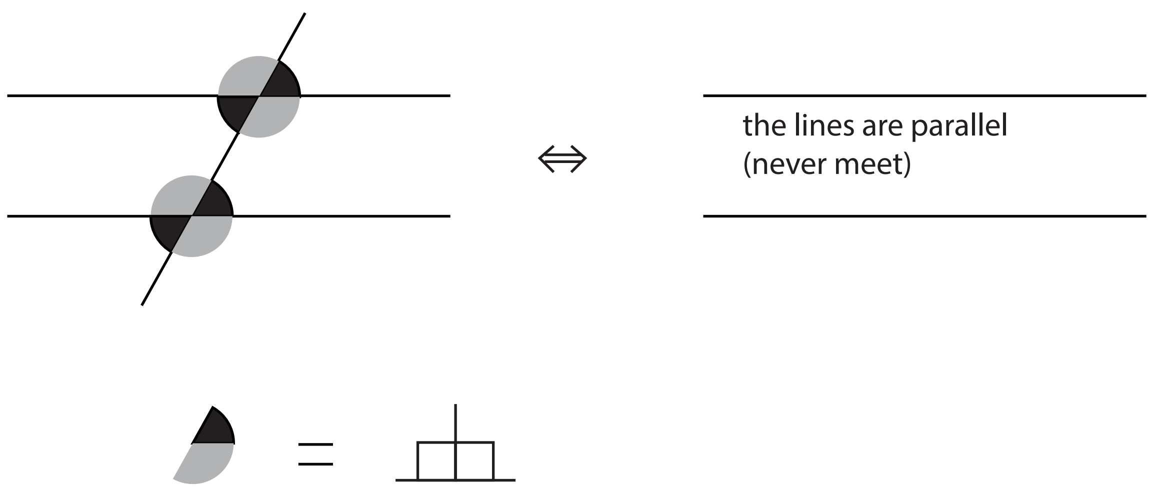
Why is SSA false? Can you give triangles ABC and DEF so that AB and DE are the same length, BC and EF are the same length, angle BCA is the same as angle EFD, but the triangles are not congruent? This is more subtle!

That just leaves ASA and AAS: Proposition 1.26.

A curious note: Euclid groups ASA and AAS together and waits until Proposition 26 to get around to proving these are true. But if you look closely, his proof of ASA could have been given any time after Proposition 4 — I wonder why he waited so long!

And then we're ready to begin talking about parallel lines! This is a big step, for it is the first time we'll be using the mysterious fifth postulate. All the theorems so far hold in any geometry that satisfies just the first four postulates: on the sphere, the plane and the hyperbolic plane. But once we use the fifth postulate, our results hold only in the plane.

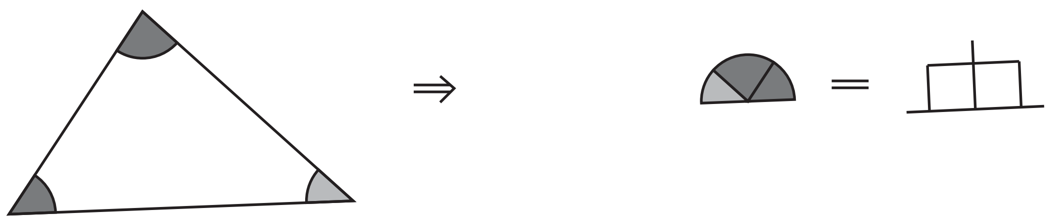
Theorems 27, 28, 29:



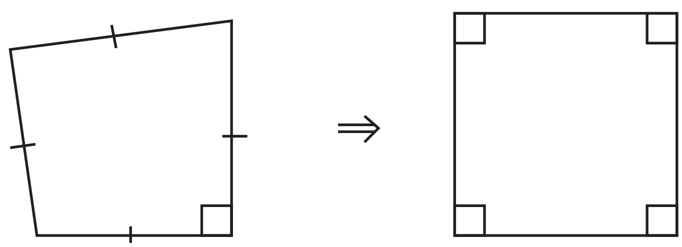
More Theorems that Require the Fifth Postulate

From these we'll be able to prove some facts that only hold in the Euclidean plane, where the fifth postulate holds. (These don't hold in non-Euclidean geometry!)

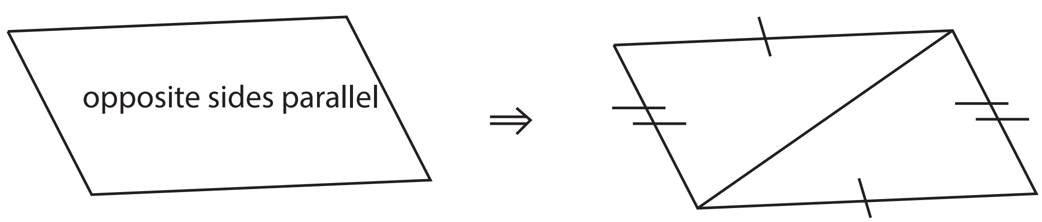
1.32 The sum of interior angles of a triangle equals two right triangles.



If all sides of a quadrilateral are equal and one interior angle is a right angle, then all angles are right angles and the quadrilateral is a square. (We needed this on the first day, when we claimed we had really constructed a square!)



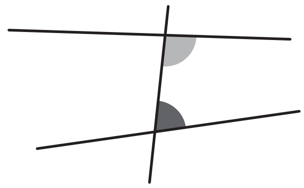
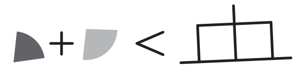
I.33 Opposite sides of a parallelogram are equal, and the diagonal of a parallelogram divides it into two equal parts.



And: I.47 The Pythagorean Theorem!!

Different possible “Parallel Postulates”

Any of these could have been taken as *the* Parallel Postulate and we would have had the same geometry.

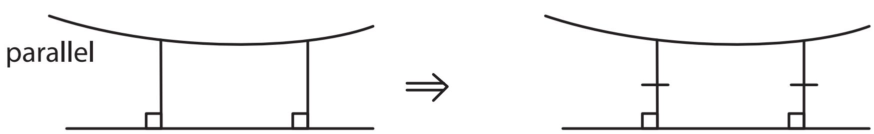
Euclid’s Parallel Postulate: Given  with 

then the lines must meet.
(Or conversely, if the lines are parallel, the interior angles are at least two right angles)

Playfair’s parallel postulate: Given

There is a *unique* line through the given point, parallel to the given line.

va: If two lines are parallel, then any two segments between them, perpendicular to one of the lines, have the same length.



Any of these can be taken as an axiom and the others proven as theorems. For example:

Taking Euclid’s postulate, prove Playfair’s and Tom’s.

Taking Playfair’s, prove Euclid’s and Tom’s.

Taking Tom’s, prove Euclid’s and Playfair’s.

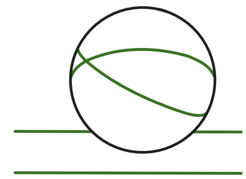
In any of these, you may use all the theorems that rest on the first four postulates, namely Propositions 1-28.

A quick word about Non-Euclidean geometry.

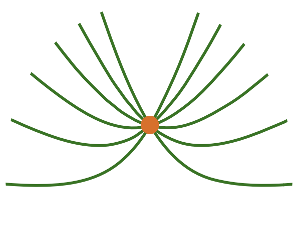
There is more than one kind of geometry in which the first four postulates hold!

The most important examples are elliptic geometry, which is the geometry of the sphere, and hyperbolic geometry.

Elliptic Geometry:
On a sphere, we can define lines to be great circles — circles that cut the sphere in half. The first four postulates work just as well, and all the theorems that depend upon them. We still have SAS, SSS, ASA, facts about isoceles triangles, and all the constructions we've seen so far.



But there are no parallel lines — every pair of great circles intersect — and we have some surprising theorems, such as: The sum of angles in a triangle is greater than two right angles.



Hyperbolic Geometry

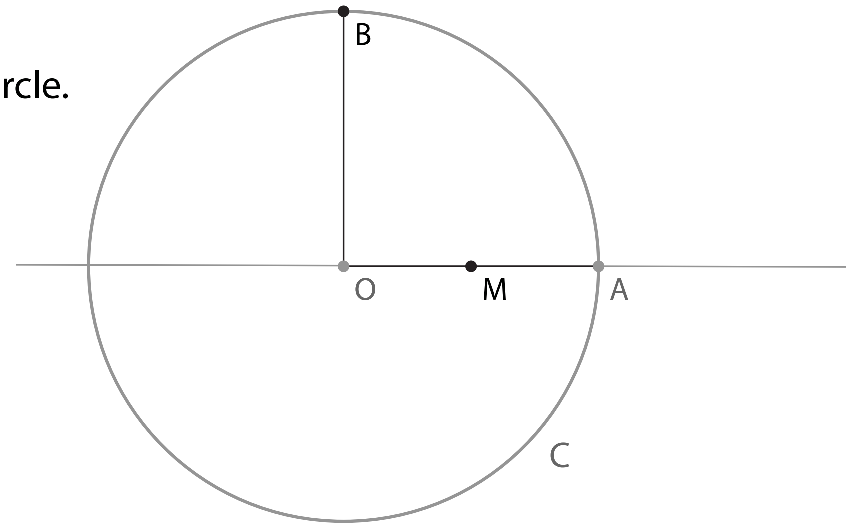
We'll need to take some time to explain hyperbolic geometry, but in essence it is the geometry of surfaces with a lot more room than in the plane — the surface of a wrinkly lettuce leaf is an example. In hyperbolic geometry, there are a great many lines, parallel to a given one, through a given point! And we have theorems like this one: The sum of angles in a triangle is less than two right angles!

One thing that holds in both elliptic and hyperbolic geometry, but not in the plane, is AAA — two triangles with the same angles have to be congruent!!

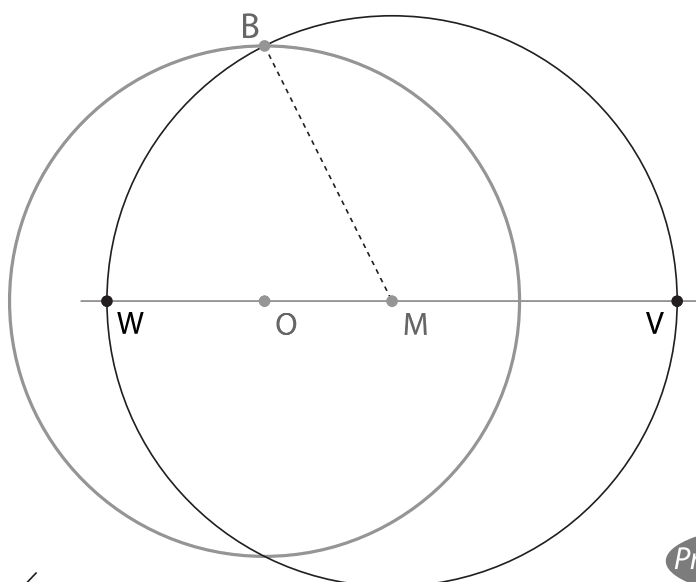
For example, can you find a triangle on the sphere with three right angles? Can tell why any two triangles like this are congruent?

Euclid's Construction of a Regular Pentagon, Given a Circumscribing Circle.

Begin with a circle C with center O. Draw any line OA passing through O. Find the midpoint M of segment OA. Draw a perpendicular radius OB.

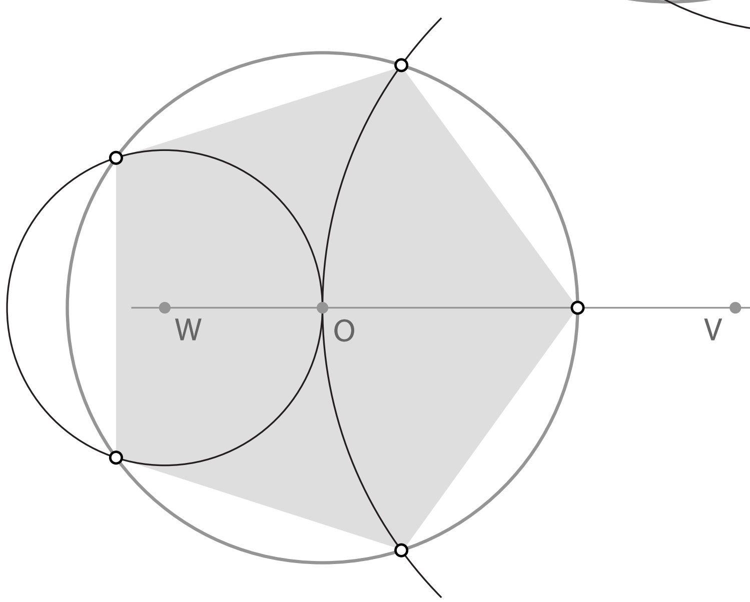


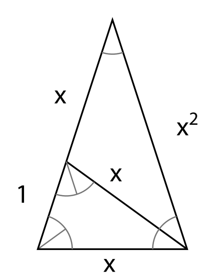
Draw the circle with center M, passing through B, and let W and V be the intersections of this circle with the line OA.



Once we have V and W, construct the circles with centers V and W passing through O. The intersections of these circles with C are four of the five vertices of the desired pentagon. The fifth vertex is our our point A.

Prove this works!



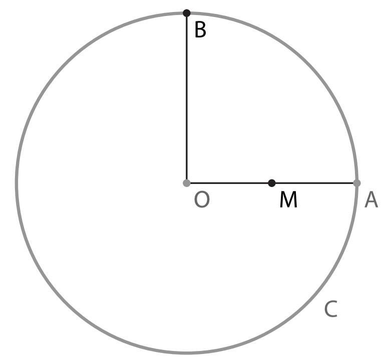


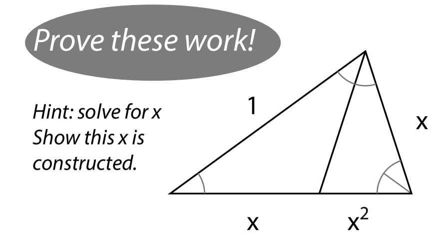
A 36-72-72° isosceles triangle has both base angles twice the top. Subdivide into two isosceles triangles as shown. Taking 1 and x as shown, by similarity we find that the sides $x^2 = 1+x$.

Solve for x. Can you see how this quantity is constructed in our construction?

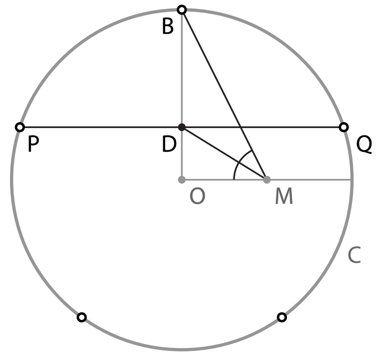
Four Constructions of a Regular Pentagon, Given a Circumscribing Circle.

All four constructions begin with a circle C with center O. Draw a radius OA and find its midpoint M. Draw a perpendicular radius OB.

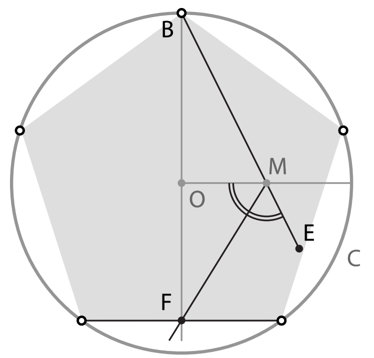




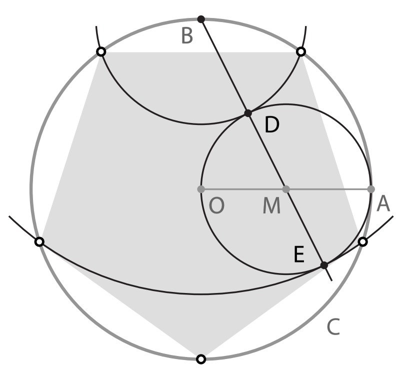
❶ Richmond's construction: construct the bisector of angle OMB and let D be the intersection of OB with this bisector. Construct the line parallel to OM, passing through D and Let P and Q be the intersections of the circle C with this parallel. P, Q, and B are three of the vertices of the desired pentagon.



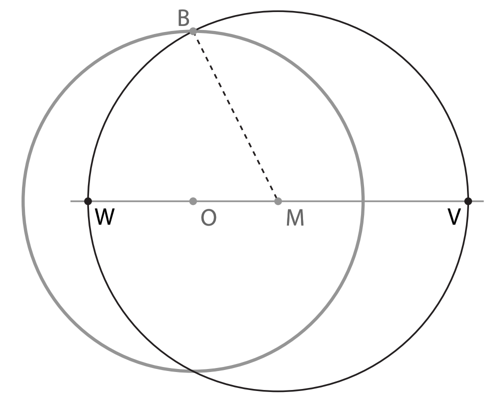
Two vertices are enough to find all the others, but it's nice to see how Richmond's construction can be continued: Extend the ray from B through M to some point E. As before, construct the bisector of angle OME, and let F be the intersection of this bisector with the ray BO. Constuct the parallel to OM passing through F—the other two vertices of the pentagon lie at the intersection of this parallel with C



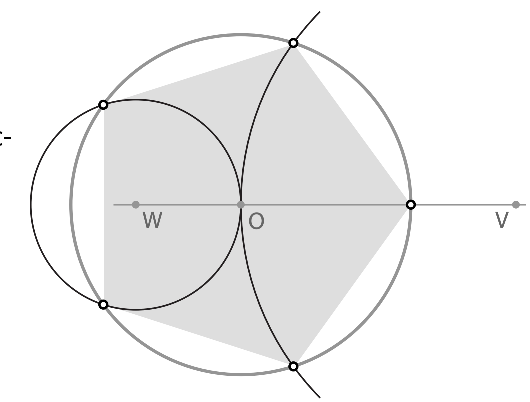
❷ Draw the ray from B passing through M, and the circle with center M passing through O and A. Let D and E be the intersections of this ray with this circle. Draw the circle with center B passing through D, and the circle with center B passing through E. The intersections of these circles with the original circle C are four of the five vertices of the desired pentagon. The fifth vertex R is opposite B on the circumference of C.



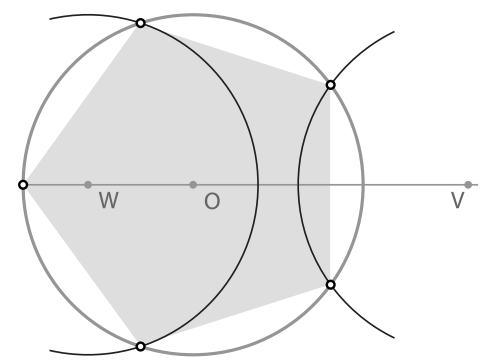
For the third and fourth constructions, draw the circle with center M, passing through B, and let W and V be the intersections of this circle with the line OA.



❸ Euclid's construction: Once we have V and W, construct the circles with centers V and W passing through O. The intersections of these circles with C are four of the five vertices of the desired pentagon. The fifth vertex is our our point A.



❹ Or we can use "Carlyle circles": construct the circles with the same radius as the circle C, but centered at V and W. The intersections of these circles with C are four of the five vertices of the desired pentagon. The fifth is the point opposite A on the circumference of C.

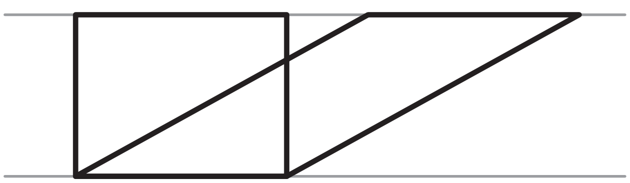


A very Cool Proof

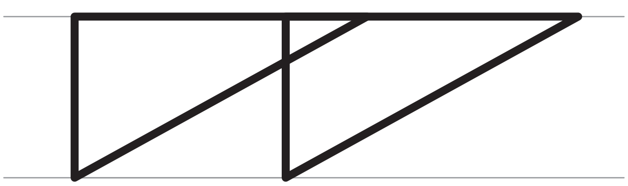
Once we prove that opposite sides of a parallelogram are equal, we can prove:

Parallelograms with equal bases, between the same parallel lines, have the same area.

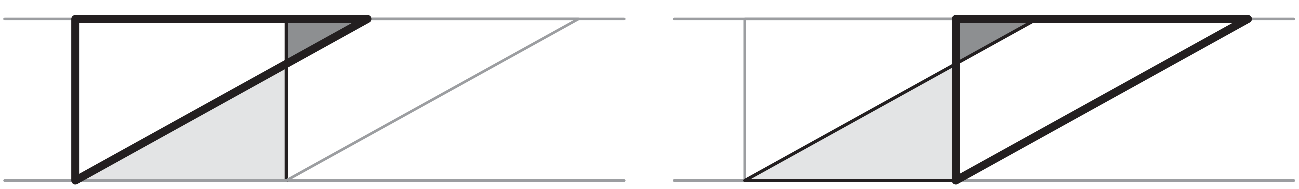
A hint of the proof. We begin with:

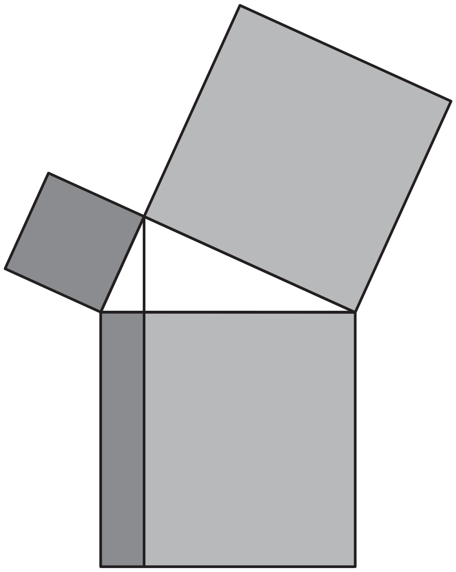


Why are these two large triangles equal?



What do we need to add and subtract from the triangles to get the area of each parallelogram?





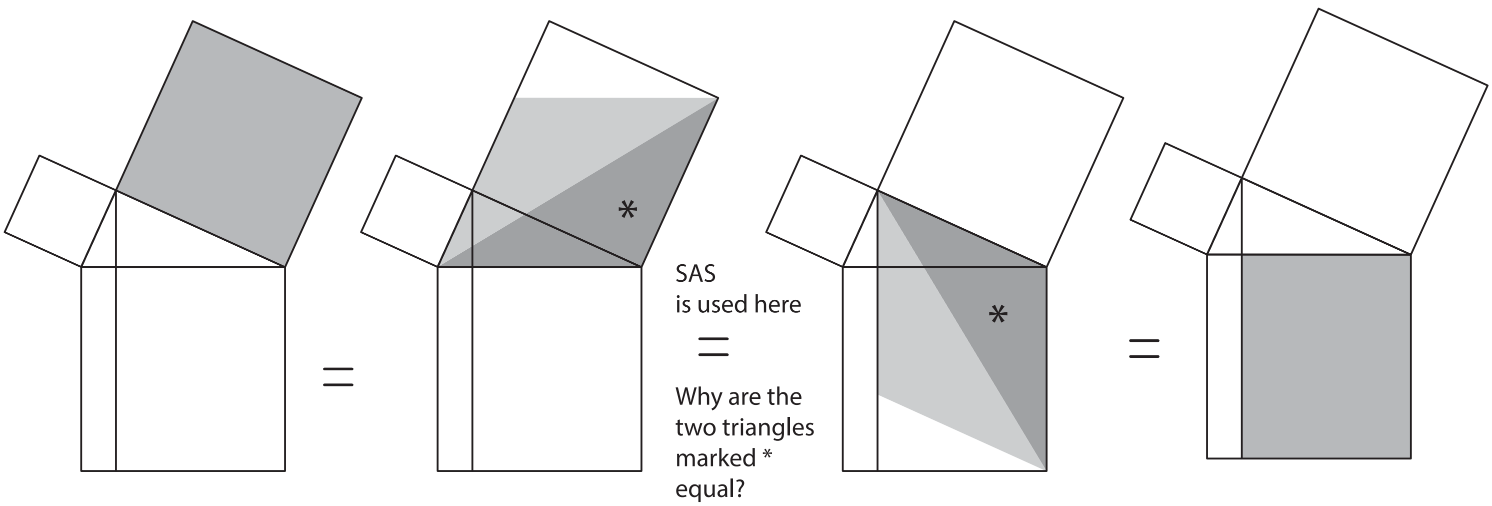
Euclid’s Proof of the Pythogorean Theorem (1.46)

We will show that the shaded regions have the same area, using three theorems:

- A diagonal bisects a parallelogram into two equal areas.
- Parallelograms with the same base and between the same parallels have the same area.
- SAS

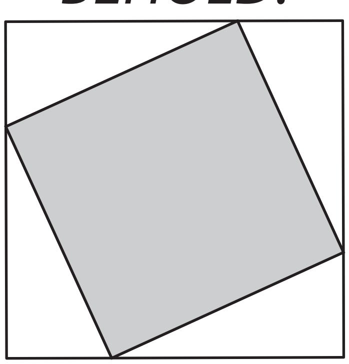
We’ll give the argument for just one of the areas: the argument for the other is exactly the same.

Why is each step valid?

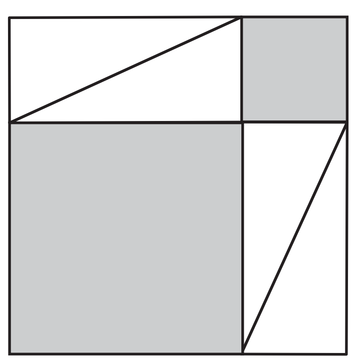


Here is another proof, known to the Chinese 2000 years ago; the Hindu mathematician Bhaskara simply wrote:

BEHOLD!

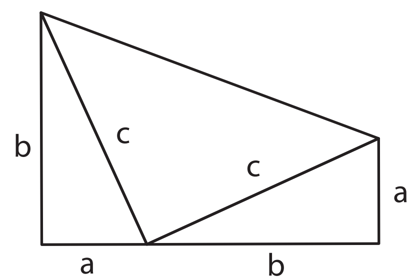


he was referring to:



Which of Euclid’s propositions are needed for this proof to be complete?

Here's a proof of the Pythagorean Theorem, due to President James A. Garfield (March 1881-Sept 1881):

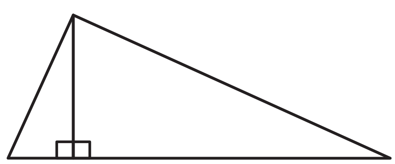


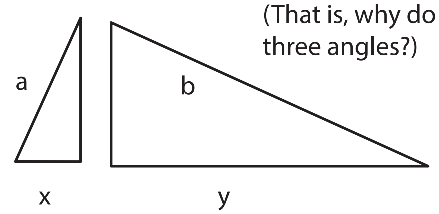
The total area of the trapezoid is ½ (a+b) (a+b).
The area of the triangles is ½ ab + ½ ab + ½ c^2
Can you use this to show $a^2 + b^2 = c^2$?

A full proof requires knowing the area of a trapezoid, which is not hard to prove using Euclid Book 1, the area of a triangle (which Euclid more or less works out in Book 1), and some algebra, which Euclid works out in Book 2.

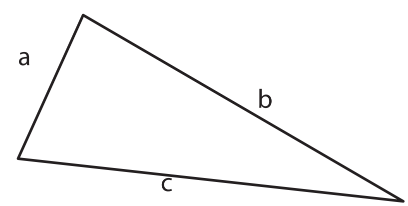
Later in the *Elements*, Euclid works out how to deal with ratios (Book V) and that corresponding parts of similar triangles are in the same proportion (Book VI). He can then give the following proof (Book VI, proposition 31)

Draw a perpendicular from the hypotenuse to the opposite vertex:





Why are all three triangles similar? (That is, why do they have the same three angles?)



It's easier for us to follow if we use our familiar notation. Explain each step:

$a / x = c / a$
Therefore $a^2 / c = x$
Therefore $a^2 / c^2 = x / c$

Meanwhile:

$b / x = c / b$
Therefore $b^2 / c = y$
Therefore $b^2 / c^2 = y / c$

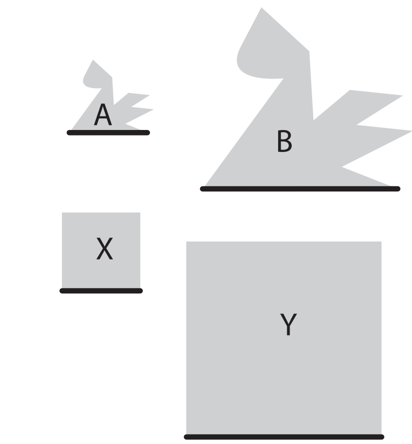
Now consider the sum $x/c + y/c$. On the one hand, $x/c + y/c = (x+y)/c$ = _______

On the other hand, substituting in, we have $x/c + y/c$ = ________

Finish the proof.

There are hundreds! See:
http://www.cut-the-knot.org/pythagoras/index.shtml

This is a stunning variation on the proof in Book VI, Proposition 31. It relies on the following fact, proven in Book VI:

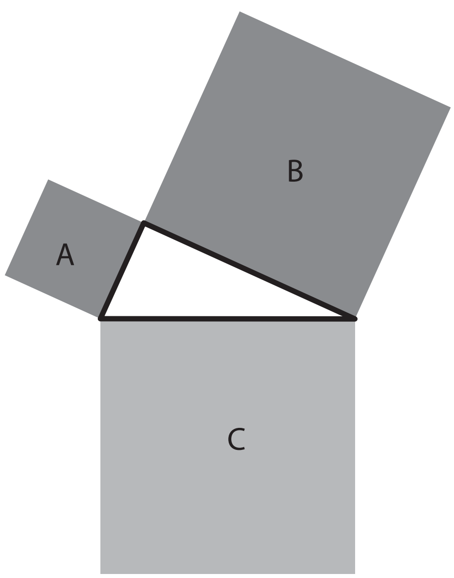


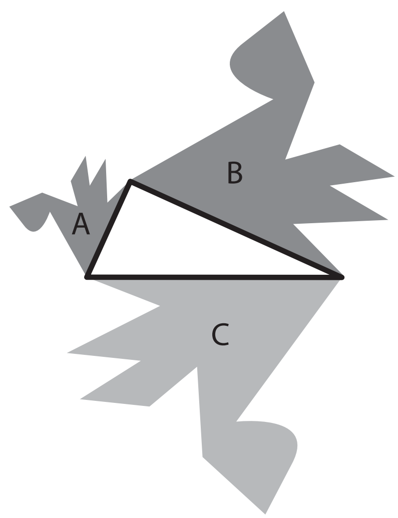
Suppose figures A and X are on the equal bases, and figures B and Y are also on equal bases.
Further suppose A is similar to B and X is similar to Y.

Then (area of A) / (area of B) = (area of X) / (area of Y)

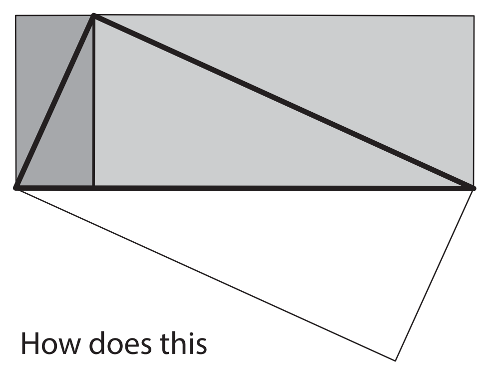
That is, the areas are in the same proportion.

What this means is that if I want the area of the two squares to add up to the area of the third, it suffices to show this for any figures that have the same bases.





SO: we might as well take something useful!



areas A+B= C $\Leftrightarrow$ areas A+B= C

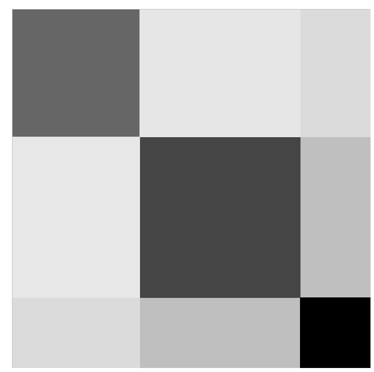
How does this complete the proof?

An introduction to Book Two

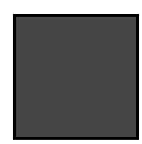
In Euclid's time, algebra had not yet been invented, and, by our standards, using algebraic facts was extremely awkward — but necessary. Book Two gives many clever geometric constructions which would today (and for the last thousand years) be considered part of algebra. For example:

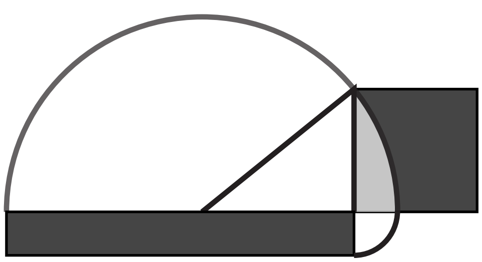
Theorem: $(a+b)^2 = a^2 + 2ab + b^2$ Proof:

Theorem: $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2ab$
Proof:

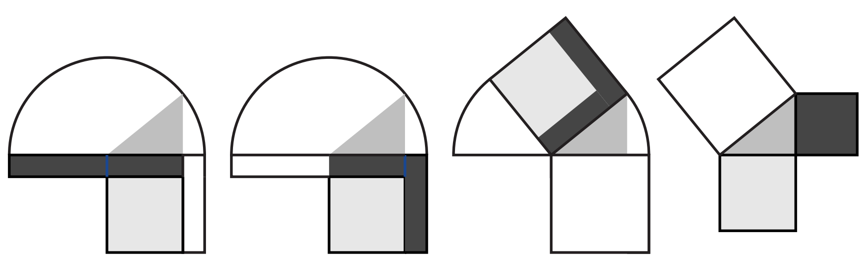


Euclid gives a clever construction of the harmonic mean:
Given a and b, construct $\sqrt{ab}$

Given  produce a square with the same area: 



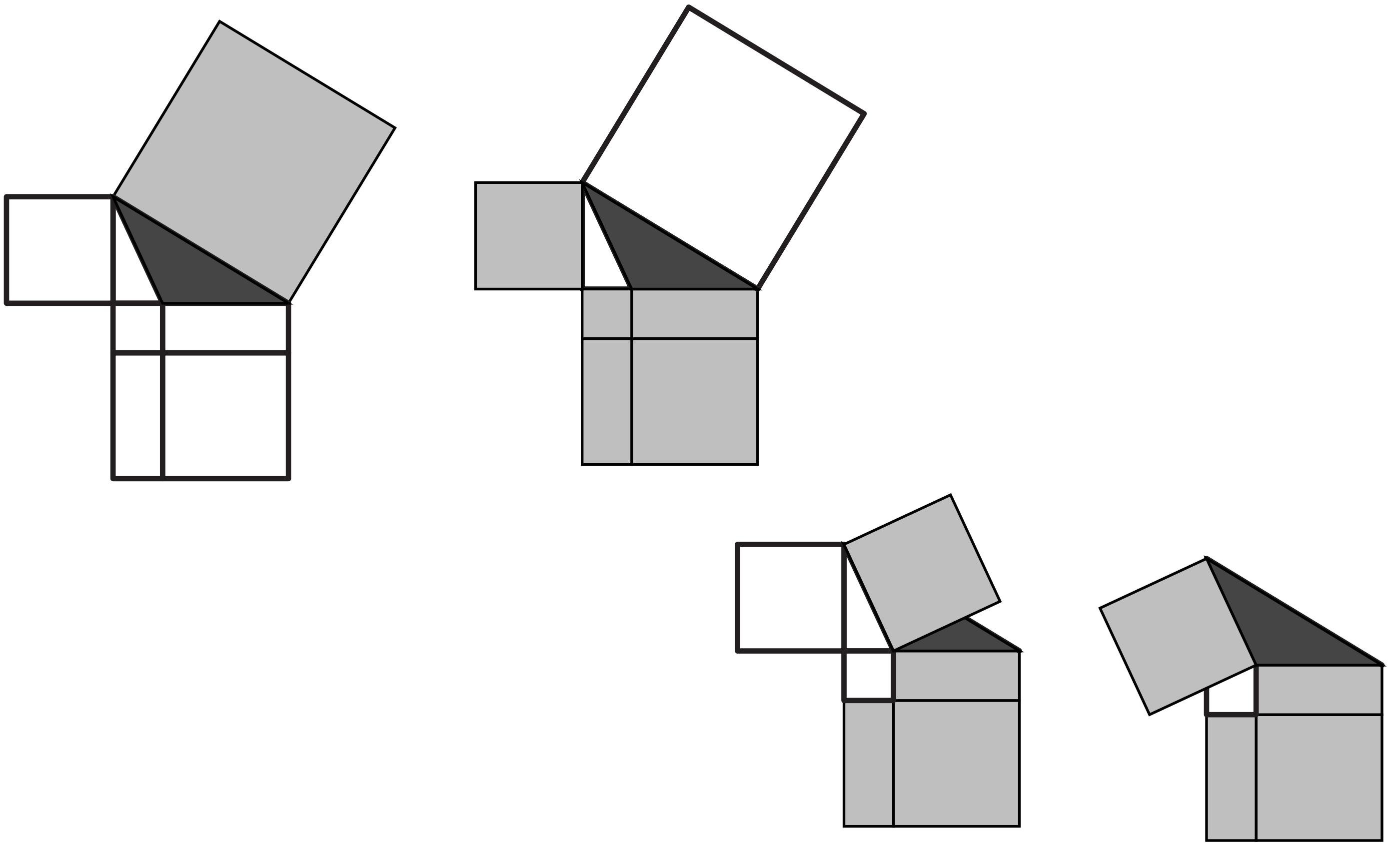
Proof:



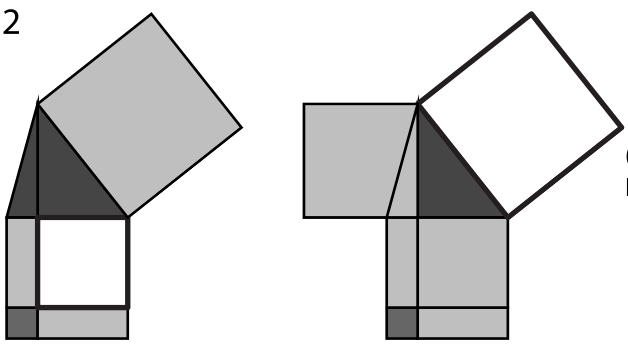
Euclid's proof of The Law of Cosines (II.13)

$a^2 = b^2 + c^2 - 2\ bc \cos A$

Case 1



Case 2

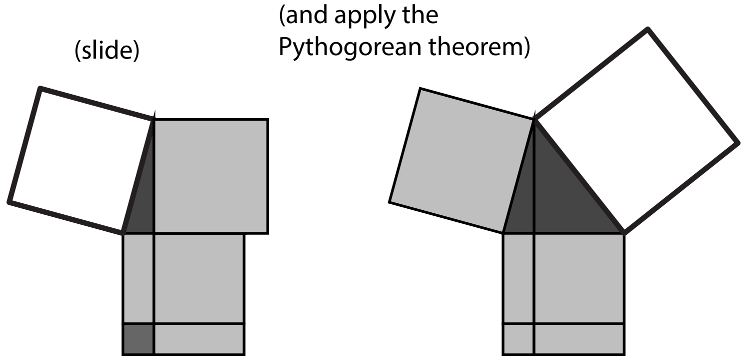


(apply the Pythogorean theorem)

$a^2 + 2\ bc \cos A =$

(slide)

(and apply the Pythogorean theorem)



$= b^2 + c^2$

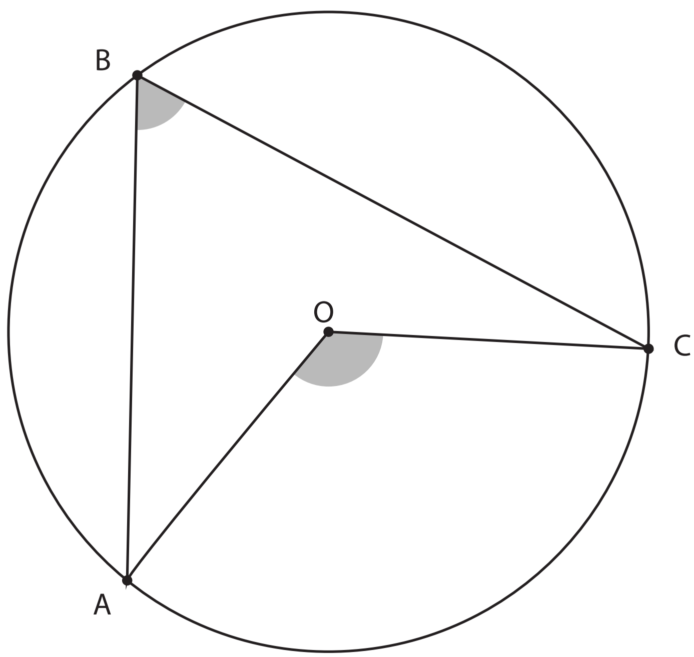
Some important facts from Book III of Euclid's *Elements*:

Construct the circle through three given points

THESE REQUIRE THE PARALLEL POSTULATE

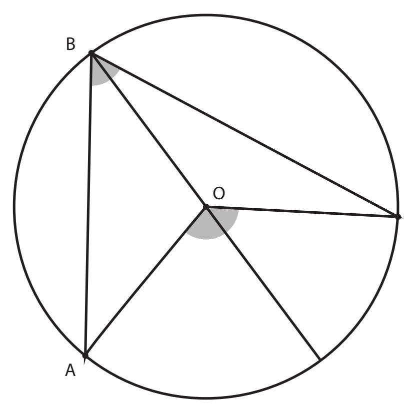
HOMEWORK
Can you supply
the proofs?

Thereom 3.20 is remarkably useful:
In any diagram like this one, angle ABC is half of angle AOC. What is the proof?
(Mainly this uses Prop 1.6, that the base angles of an isosceles triangle are equal.)



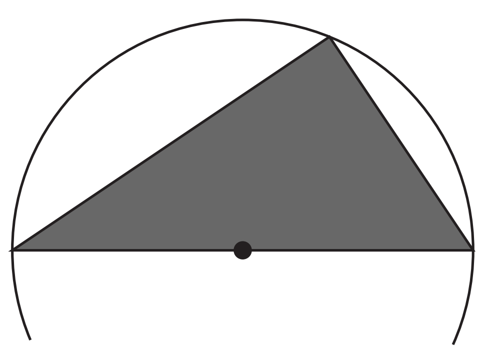
HINT: Split into two parts using the diameter BO. Look at each part separately.

Do you see a lot of isosceles triangles?



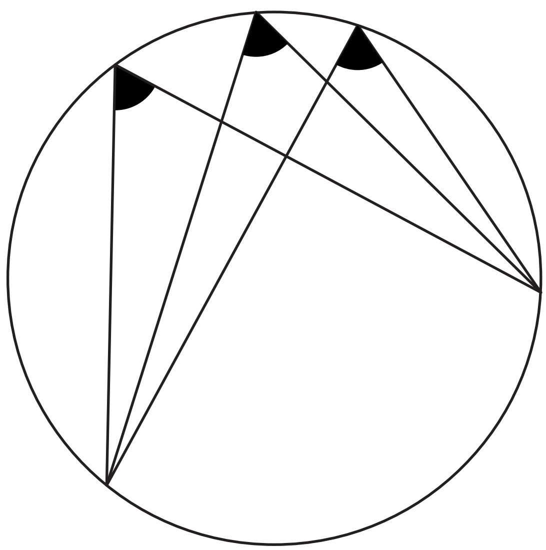
This theorem has some remarkable corollaries, such as:

Corollary: Any triangle with vertices on the circle, and one side a diameter of the circle, must be a right triangle!

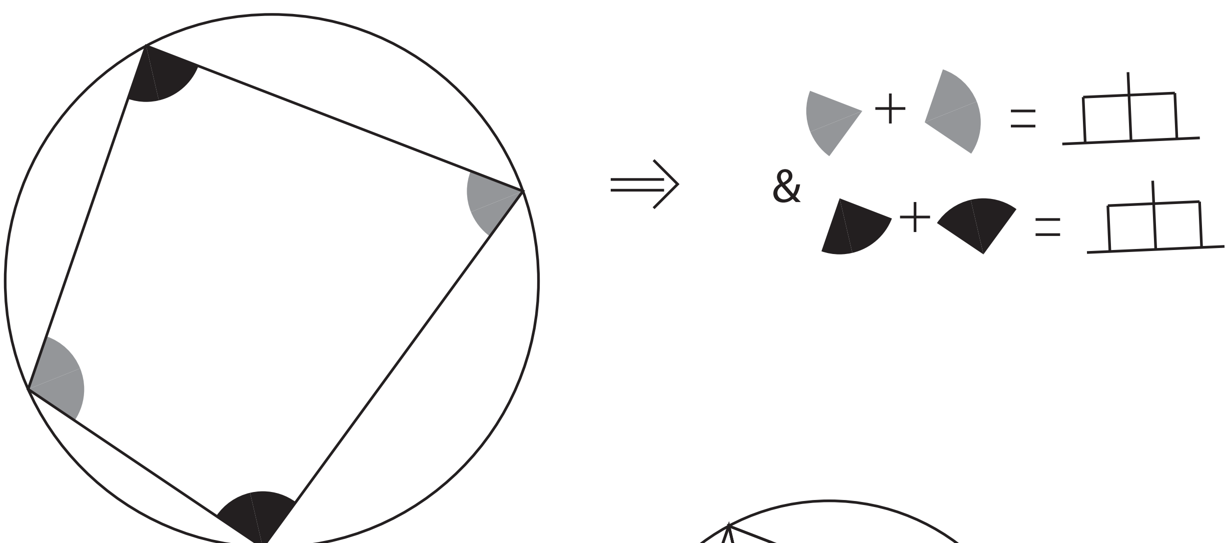


Corollary (3.21) Incredibly, if angles on a circle subtend the same arc, they must be equal.

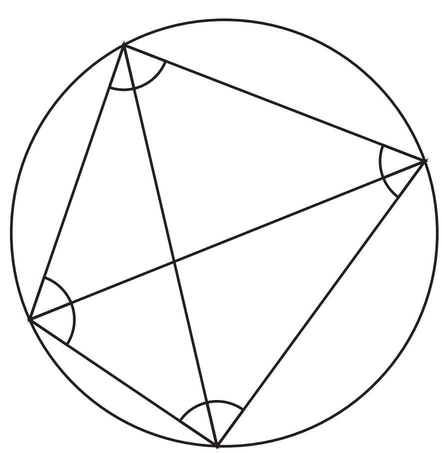
HOMEWORK
Can you supply the proofs?



Corollary (3.22) If all the vertices of a quadrilateral lie on a circle, then opposite angles must sum to two right angles.



Hint: Use this diagram,
Theorem 3.20,
and the basic fact that
the sum of the internal angles
of a triangle equals two
right angles (1.32).



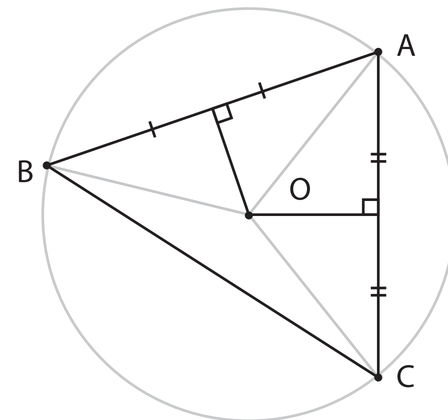
Some facts about circles and triangles.

We have already seen how to construct the circle through three given points: (Euclid IV.5) Construct the perpendicular bisectors to two the legs of the triangle, the point where they meet will be the center of the triangle.

But why does this work? Given points ABC, construct the perpendicular bisectors to AB and AC, and let them meet at point O. (Why must they meet at all?)

Then we must show that OA, OB, and OC are all of equal length. But that's just a few applications of Hypotenuse-Leg (which follows from the Pythagorean Theorem.)

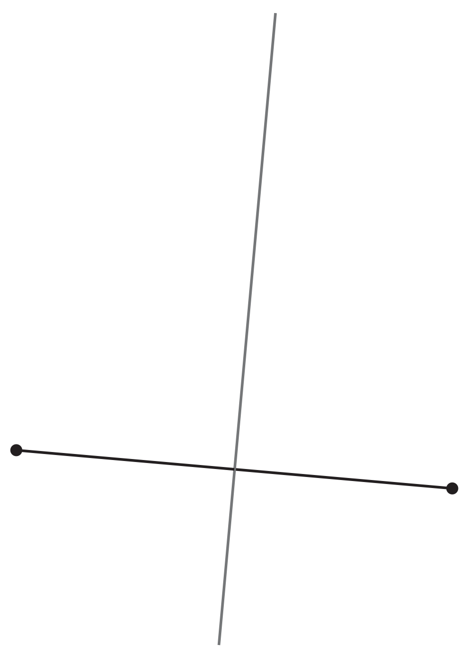
Thus, the circle centered at O, passing through one of the vertices of the triangle, also passes through all three.



Note that this implies, and is implied by:

Every circle through any two given points lies on the perpendicular bisector of the segment joining them.

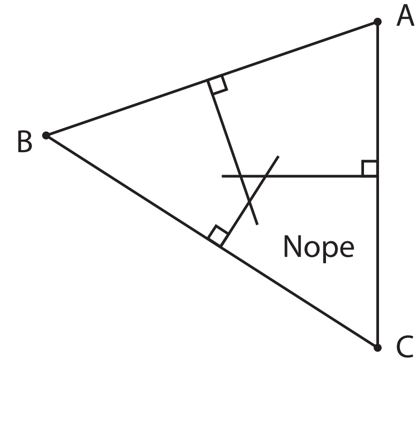
Try this:



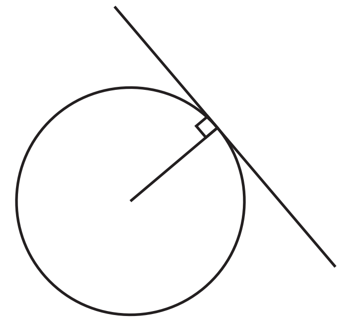
Draw a bunch of circles on the perpendicular bisector of this segment, passing through both ends.

I think this is a pretty cool picture!

The construction also implies an interesting thing: the perpendicular bisectors must all meet at a point (the center of this circle). Initially, it's not clear that the perpendicular bisectors couldn't look something like this instead:

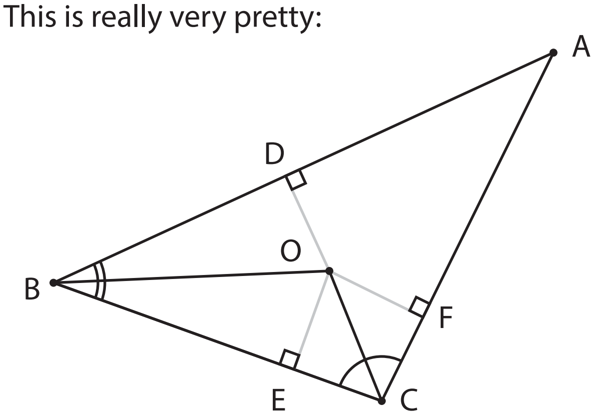


Euclid III.16 tells us that any line tangent to a circle at a given point must be perpendicular to the radius of the circle at that point.



We'll use this for our next construction.

Let's construct the circle inscribed in a given triangle (IV.4). This is really very pretty:



Construct the angle bisectors of angles ABC and ACB.

Let O be the point where these meet.

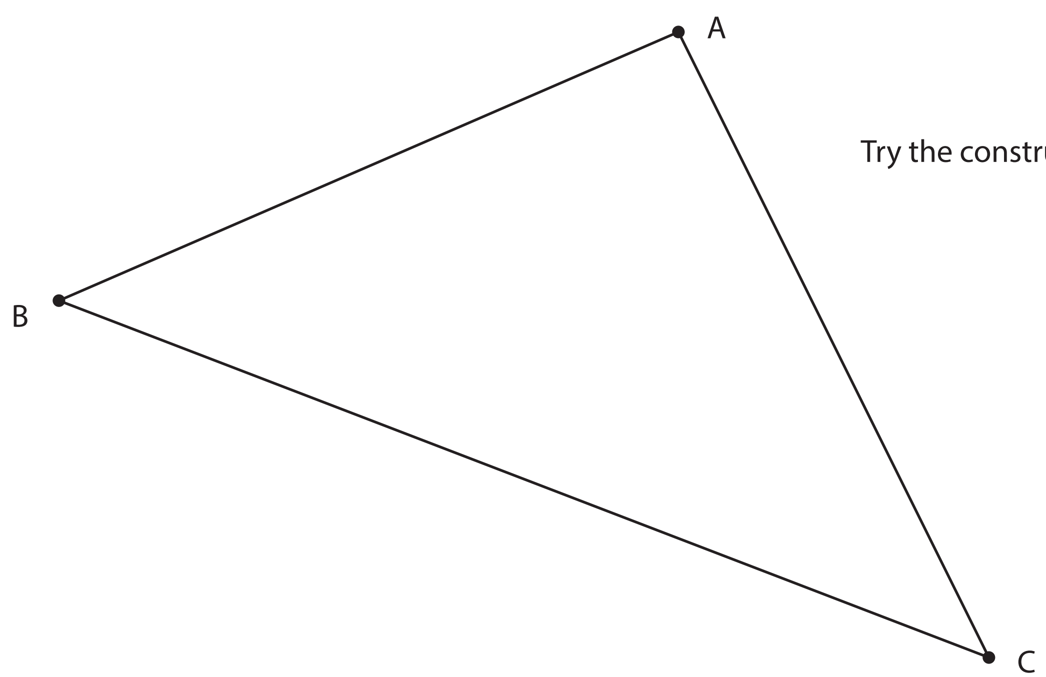
From O, construct the perpendiculars to each of the segments, meeting at points D, E, and F.

The claim is that segments OD, OE, and OF are all equal.

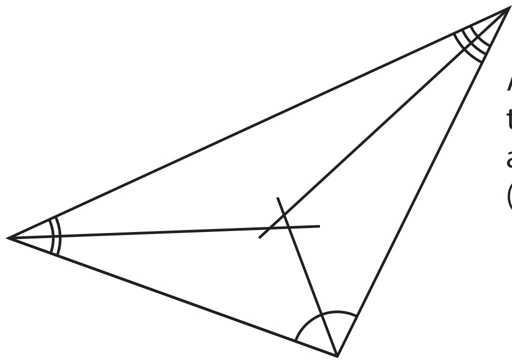
Why is this true?

So the circle centered at O, passing through one of the points D, E, or F, passes through all three of them.

Why must this circle be tangent to the sides of the triangle?



Try the construction yourself.

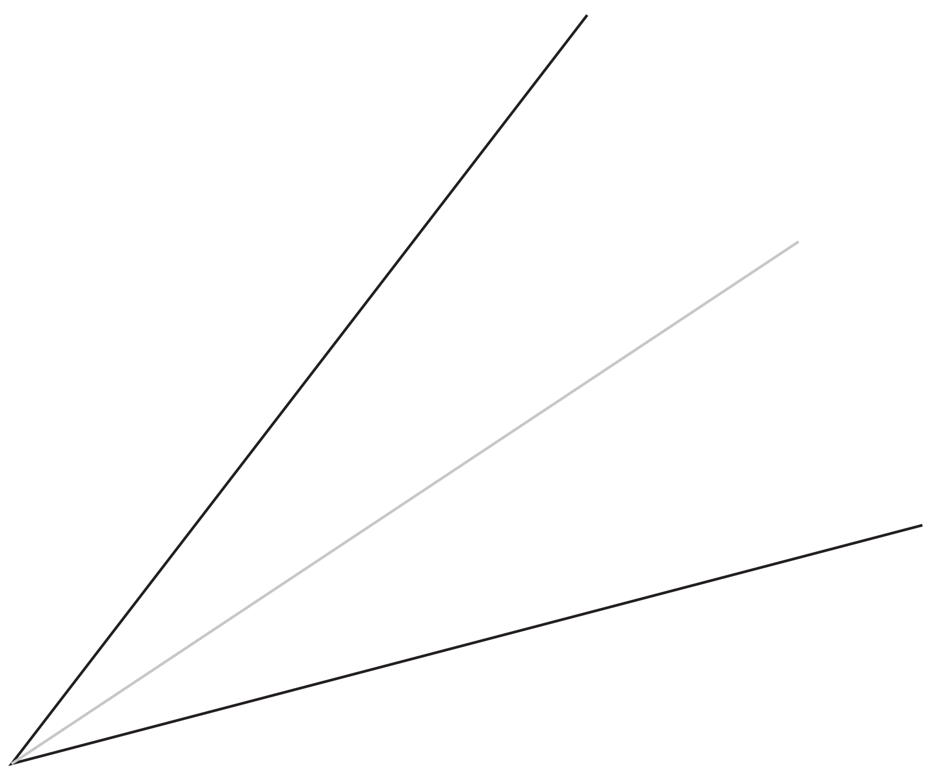


And we get another interesting observation: This implies that all the bisectors of the interior angles must meet at a single point. We can't have a picture like this one. (Why?)

This also implies, and is implied by, the following observation: If a circle is tangent to two lines that meet, its center must lie on the angle bisector.

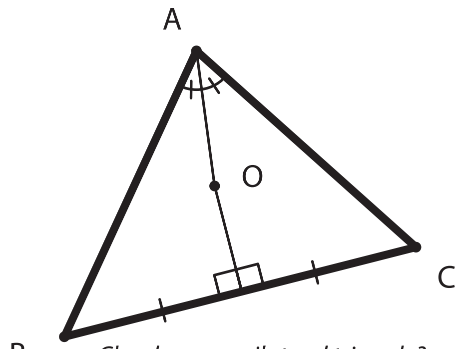
Once we choose a center, though, we must find the perpendicular from the center to one of the sides to to find a point on the circle itself, to get the right radius.

Try it!



A "proof" that every triangle is equilateral.

It is remarkably difficult to spot the error — If you make a careful drawing, you will be able to discover where the flaw is. Hint: all the triangles we will claim are congruent really are — that's not the problem.

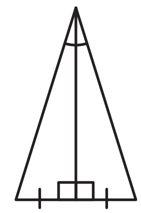


Clearly, an equilateral triangle?

Given any triangle ABC, we shall prove that AB is congruent to AC. Consequently, since we may label the vertices however we please and the proof will still "work," this implies that all three sides are congruent and the triangle is equilateral!

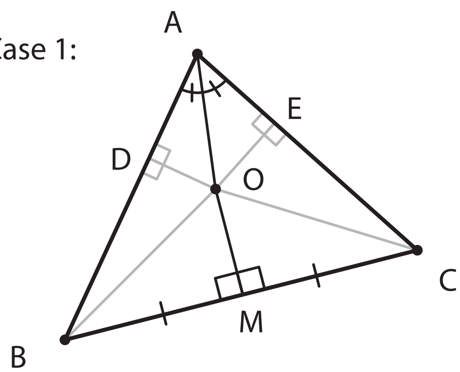
Begin by constructing the angle bisector of angle BAC and the perpendicular bisector of segment BC. Suppose for contradiction that triangle ABC is not isosceles. Then these lines are not parallel and meet at some point O.

Clearly O cannot be on segments AB or AC (since O lies on the bisector of the angle between them). And if O lies on segment BC, the angle bisector is then the perpendicular bisector of BC, and the triangle would be isosceles (AAS). In that case we have proven AB is congruent to AC as promised.

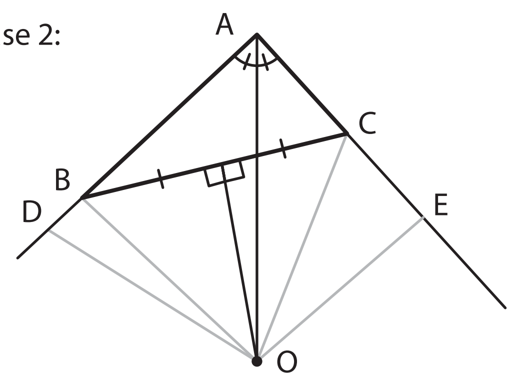


We still have two cases to consider: The point O is inside the triangle ABC, or the point O is outside the triangle B.

Case 1:



Case 2:



Let M be the midpoint of BC.
From O drop the perpendiculars to sides AB and AC, to points D and E, and draw segments BO and CO.

Now, triangles BMO and CMO are congruent, by SAS, so segments BO and CO are congruent.

Triangles DOA and EOA are congruent by AAS, so segments DO and EO are congruent, as are segments AD and AE.

Triangles DOB and EOC are both right triangles, and have congruent hypotenuses and a congruent leg. By the Pythagorean theorem the other legs, segments BD and CE are congruent as well, and so by SSS, are congruent triangles.

Segment AB is equal to AD and DB. Segment AC is equal to AE and EC. Since segment AD is congruent to segment AE, and segment DB is congruent to segment EC, we have proved that segment AB is congruent to AC.

Let M be the midpoint of BC.
From O drop the perpendiculars to lines AB and AC, to points D and E, and draw segments BO and CO.

Now, triangles BMO and CMO are congruent, by SAS, so segments BO and CO are congruent.

Triangles DOA and EOA are congruent by AAS, so segments DO and EO are congruent, as are segments AD and AE.

Triangles DOB and EOC are both right triangles, and have congruent hypotenuses and a congruent leg. By the Pythagorean theorem the other legs, segments BD and CE, are congruent as well, and so by SSS, are congruent triangles.

Segment AD is equal to AB and BD. Segment AE is equal to AC and CE. Since segment AD is congruent to segment AE, and segment DB is congruent to segment EC, we have proved that segment AB is congruent to AC.