The Unsolvable Configuration of the 15 Puzzle and an Interesting Approach to Abstract Algebra

By Kareem Jaber

Math, to a lot of people, is all about numbers. We've spent all of our time in school learning how numbers work, how to add them and subtract them, how to multiply and divide them. We've learned how they interact with each other in equations, seen how they can transform each other with functions, and a lot, lot more. After all, numbers are the language of math, right?

But what if they aren't? What if we threw away our preconception of what numbers are and what we can do with them, and instead look at something more fundamental? The field of abstract algebra is exactly this: we look at the structure of how systems interact. This way of thinking gives us access to solving lots of interesting problems that "normal math" can't solve, and today, we're going to be using some abstract algebra to play with a fun little toy: the 15 puzzle.

This isn't so much a paper about abstract algebra, it's more of an opportunity for me to share a playful math experience with you. You shouldn't try to read this like a book, but instead use this as a chance to work through a problem, to struggle a little bit, but to enjoy the struggle, and to appreciate the art of curiosity, motivation, and problem solving.

Alrighty, the 15 puzzle is the puzzle to the right with, unsurprisingly, 15 sliding pieces labeled 1-15. When the puzzle is mixed up, the goal is to push the pieces around to try to reorder the pieces. If you have one at home, go pull it out, but if not, try out this online version at https://tinyurl.com/fifteenpuzzleonline (credit to Simon Fraser University). Mix it up, try to solve it, and get an idea of how pieces move. Break it down into steps.

This is how I go about solving the 15 puzzle:

Step 1: Get piece 1 in position, then piece 2. Pieces 3 and 4 have to go in simultaneously because of the way pieces move, so place the two pieces next to each other and use a "snake-like pattern" to put them where they need to go (check out the picture to the right). After this row is complete, don't touch it again.

Step 2: Follow the same procedure to get pieces 5 through 8 into their positions. After this row is complete, don't touch it again.

Step 3: Get pieces 9 and 13 in position, then don't touch them. Get pieces 10 and 14 in position, and then the 11, 12, and 15 pieces should be easy to solve.

Solve the puzzle a few times. Try to see how fast you can do it. My record is a bit under 20 seconds!

Here's my real challenge. Let's say that I took your puzzle (that you just solved) but shuffled it a little differently than normal: I broke your puzzle, took out the 14 and 15 pieces, swapped them, and put them back. Now your goal is to solve it.

[Note: if you're using the online version, you can't physically switch the 14 and 15 pieces. Instead, try going from the solved state to the configuration where the 14 and 15 are switched.]

I'll wait. Go try it.

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If you're having some difficulties, try for a little longer.

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Hmm. I'm going to take a guess and say that you didn't solve it yet. You should feel a little bit suspicious about this situation. Something seems off.

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You still haven't solved it? "Maybe it's impossible!" you might say. Who knows? Maybe it is possible, or maybe it isn't, but you've just made a big step in the problem solving process: you've made a claim. You're arguing that the challenge I just gave you, solving this configuration with the 14 and 15 pieces swapped, is impossible. Now our goal is to prove this claim. But where are we going to start? Will the solution just magically pop into our heads? In problem solving, proving a claim requires **motivation**. We need to find a foothold to push off of, something tangibly different that we can notice behind the **structure** of the solved configuration and the 15-14 (shuffled) configuration. Take some time to play around with the 15-14 configuration and focus on what feels fishy about the situation at hand.

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How about this, let's try to take a look at a configuration similar to this: swap the 13 and 14 pieces, and then swap the 14 and 15 pieces, like this. Can you solve this configuration? If so, why?

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Keep trying this out. Try seeing if you can solve some other shuffled combinations, like these:

Left: swap 12 with 15

Middle: swap 9 with 10, and 11 with 12

Right: change 9-12 row to 12, 9, 10, 11

Which ones are solvable? What about their **structure** makes them different than the ones that can't be solved? Do certain shufflings "collapse" into others? Keep exploring and making observations.

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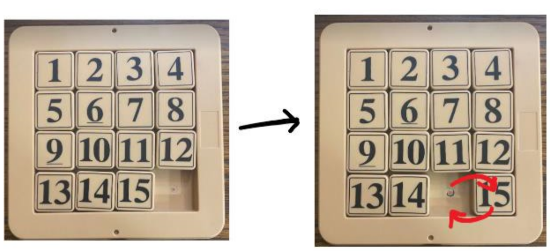
The structure of the 15 puzzle seems to be intrinsically built upon "moves," the action of swapping the empty space with a piece next to it. Can we extend this concept to our weird shufflings of the puzzle by allowing illegal moves, switches that swap any two pieces, not just the empty space and an adjacent piece? What we are doing here is another powerful tool in mathematics: we are generalizing a concept to describe the things we see around us. Let's create a new concept that allows us to extend the idea of a move and describe the nature of our shufflings:

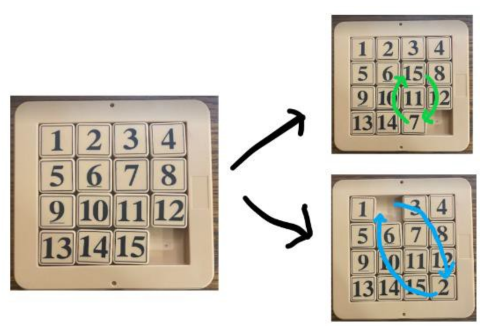
- Let's call a **move** any action that is allowed in the normal 15 puzzle game: changing the positions of the empty space and a piece next to it.
- Let's call a **switch** any action that swaps any two pieces or swaps any piece with the empty space. Notice how all switches are moves, but not all moves are switches, kind of like how all squares are rectangles, but not all rectangles are squares. Switches are our generalized concept of moves.

Here are some examples for us to get an idea of the terms we defined:

This is a move **and** a switch.



These are switches, but **not** moves.



We can describe the 15-14 configuration by starting with the solved configuration and one switch between the 14 and 15 pieces. For some reason, we can't seem to find a sequence of moves to bring the 15-14 configuration to the solved configuration. Why? What is intrinsically different about a move and a switch in terms of their structure? What always changes when you make a move? How about with a switch? What sometimes changes? What stays the same?

Again, take some time to think about this before moving on.

One thing moves always change is the position of the empty space. Every time you make a move, you are swapping the empty space with a piece next to it. Switches, however, don't always have to move the empty space, and don't always have to move a piece with one next to it. The structure of moves seems very restricted, while the structure of switches seems very loose.

Let's focus on the empty space because it seems very fundamental to the structure of moves. The position of the empty space doesn't move between the solved configuration and the 15-14 configuration. If we assume we can get from one configuration to the other by only using moves, how many moves would we need? How many moves do we need to bring the empty space back to where it was in the beginning?

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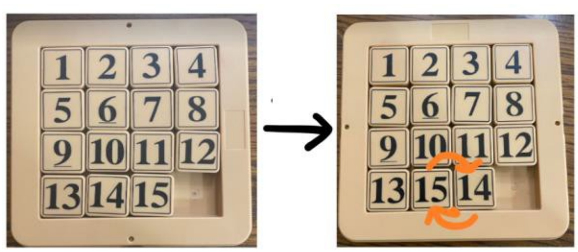
Every time we swap the empty space with the piece above it, we're eventually going to have to swap the empty space with the piece below it to "cancel out" the move up. And every time we swap the empty space with the piece to the left of it, we need to swap it with the piece to the right of it to "cancel out" the move left. This means that moves come in pairs, that we must make an **even** number of moves to bring the empty space back to where it was before. This means that hypothetically, if we could go from the solved configuration to the 15-14 configuration, it would take an **even** number of moves.

The same is not true for switches. We can make one switch, an **odd** number of switches, to get from the solved configuration to the 15-14 configuration, just like this:



We've finally found a foothold, a hint of motivation: **parity** (the math-y word for oddness and evenness) is fundamentally different for moves and switches. While it takes an even number of moves to leave the empty space untouched, switches do not have this restriction.

Look at some of the other "shufflings" that we took a look at earlier. Keep track of how many switches it takes to shuffle them, and whether or not the empty space was moved in the process.

Think about the parity of the number of moves it would (hypothetically) take to bring the shuffled configuration to the solved configuration.

We've made a lot of observations. We've explored the structure of the 15 puzzle and found some motivation to start attacking our problem. Let's do a quick recap:

- We think that the 15-14 configuration is unsolvable using only moves
- We think that it has something to do with the structure of moves and switches
- We found some interesting differences in parity between moves and switches

With our newfound motivation, we'll try to reword our original question using our new terms and come up with a more general statement to prove. If you analyzed the other shufflings from earlier, you might have noticed that the shufflings with an odd number of switches seem unsolvable for the same reason as the 15-14 shuffling: we need an even number of moves to keep the empty space in the same place.

With this, let's make a leap of faith that might seem out of nowhere. We're going to claim:

If you can go from the solved state to a configuration in the 15 puzzle with an odd number of switches, then you can't get to the same configuration from the solved state with an even number of switches.

We don't know if this is true, but we definitely have some reasonable suspicion to believe that it is. Whenever we have a shuffling requiring an odd number of switches and leaving the empty space where it is, we can't seem to find an even number of switches to bring the puzzle back to the solved state.

Also, make sure you see that proving this claim also proves that the 15-14 configuration is unsolvable. Shuffling the puzzle to the 15-14 configuration requires an odd number of switches, while solving the configuration using only moves requires an even number of switches (remember, all moves are switches). Again, reread this until you are comfortable with our statement and understand where this seemingly random leap of faith comes from.

This claim is a big one, and something that doesn't look easy to prove. So just like earlier, let's explore a bit and simplify the problem down to its core. As we do that, try to look ahead and build a game plan on how our proof will go.

Think back to our earlier findings. We are still working with the structure of the 15 puzzle just like we were doing earlier, so we don't want to forget what was important to us. One of the things that stood out the most was the concept of parity and keeping the empty space in the same place. Let's dig deeper into the root of this problem by looking at the parity of the **action that does nothing**. If we start in the solved configuration, perform a bunch of switches, and end up back at the solved configuration, how many switches did we make? Was it an odd or even number of switches?

Play with this for a while. Try to get a feel for this concept and find some patterns.

Clearly, we can perform the action of doing nothing, which we'll call the **identity** action, with zero switches. We can also do the identity action with two switches (by repeating the same switch twice), four switches (by repeating the same switch four times), six switches, and any even number of switches. We can never do the identity action with one switch, since the two pieces we switched would not be in their original places, and it would be reasonable to guess that we can never perform the identity action with an odd number of switches. But before we go through the trouble of showing this, let's think about what would happen if this statement was true.

Take a few minutes to think through how we could use this fact, that the identity action can only be done with an even number of switches, to progress through this problem. How can we connect the identity action back to the 15-14 configuration?

When playing with the identity action earlier, you might have realized that finding a series of moves that do the same thing as "doing nothing" feels a lot like shuffling the puzzle and then solving it. Assuming that the identity action can only be done in an even number of swaps (the thing that we skipped proving for now), let's try applying this idea to the 15-14 configuration.

We know we can get to the 15-14 configuration from the solved configuration in one swap (an odd number of swaps): just swap the 14 and 15 pieces. Now, we know that if it is possible to solve the 15-14 configuration using only moves, it would require an even number of moves, or equivalently, an even number of switches (because all switches are moves).

It took an odd number of switches to shuffle the puzzle to the 15-14 configuration, and an even number of switches to solve the puzzle. Since an odd number plus an even number is still an odd number, this means that **we just did the identity action with an odd number of switches.** This is HUGE. Take a second to let that sink in and realize the power of that statement.

We contradicted ourselves! We assumed that the identity action could only be done with an even number of switches, and then showed that if the 15-14 configuration can be solved using only moves, the identity action could be done with an odd number of switches. This means one of two things:

The identity action can be written as an odd number of switches
OR
The 15-14 configuration can't be solved by only using moves

So, if we can actually prove that the identity action cannot be written as an odd number of switches, then the 15-14 configuration **has** to be unsolvable! I think that this is a beautiful line of logic, so take a second to appreciate what we did together.

Homestretch! Let's show that the identity action can only be written as an even number of switches, and never as an odd number of switches. This is a really neat argument, so check this out:

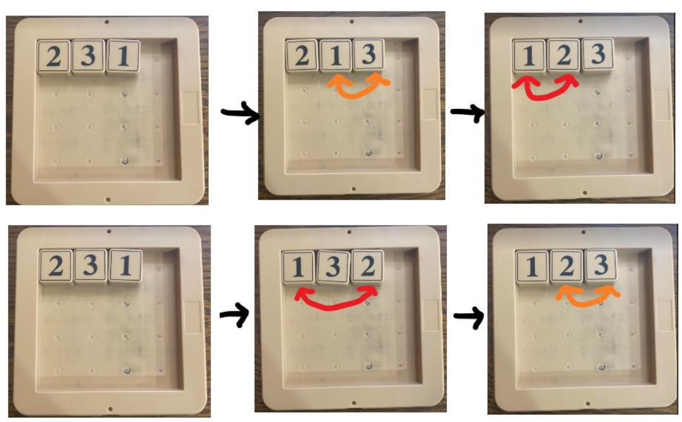
Let's say I started with a solved puzzle, closed my eyes, made a bunch of switches, opened my eyes, and saw that the puzzle was still solved. I don't know how many switches I made, or if it's an odd or even number of switches, and neither do you, so let's just give it a letter placeholder: N. I made N switches and the solved puzzle remained solved. I argue that I could have done this in N-2 switches.

To show this, let's keep track of a piece. I'll keep track of the "1" piece, but it doesn't really matter what piece we keep track of. Somewhere in our N switches, we'll switch the "1" piece for the very last time, let's say with the "2" piece (but again, it could be any piece). What could have been the switch that happened right before switching 1 with 2? There are four possibilities:

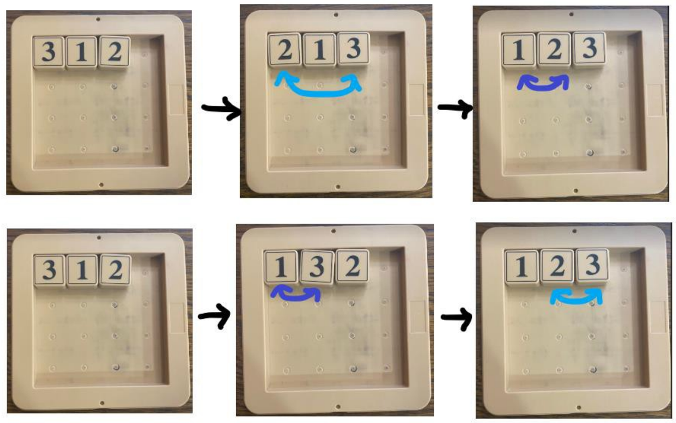
- CASE 1: I could have switched 1 with 2 again
- CASE 2: I could have switched 1 with any other number but 2, which we'll just say is the "3" piece
- CASE 3: I could have switched 2 with any other number but 1, which we'll just say is the "3" piece
- CASE 4: I could have switched any other two numbers, which we'll just say are the "3" and "4" pieces

CASE 1: If we switch 1 with 2, and then immediately switch 1 with 2 again, the switches "cancel" each other out, so we could get rid of these two switches and be left with N-2 switches.

CASE 2: If we switch 1 with 3, and then switch 1 with 2, we could do this instead: we could switch 1 with 2, and then switch 3 with 2. This means we touch the "1" piece for the last time one switch earlier. We'll come back to this in a second.



CASE 3: If we switch 2 with 3, and then switch 1 with 2, we could do this instead: we could switch 1 with 3, and then switch 2 with 3. Again, this means we touch the "1" piece for the last time one switch earlier.



CASE 4: If we switch 3 with 4, and then switch 1 with 2, we could just do them in reverse order: switch 1 with 2 and then switch 3 with 4. We touch the "1" piece for the last time one switch earlier.

For CASE 2, CASE 3, and CASE 4, let's run this casework again, but this time, the "1" piece is touched for the last time one switch earlier. Again, either:

CASE 1: the last switch where we touch the "1" piece and the switch before that are the same, so they cancel each other out, getting rid of two switches and letting us do the identity action in N-2switches

CASE 2-4: the last switch where we touch the "1" piece and the switch before are not the same, and we run our argument again and touch the "1" piece one switch earlier.

We can keep doing this over and over and over again, until we end up at CASE 1. But what if we never get to CASE 1? This means that we keep moving the last time we touch the "1" piece until the last and only time we touch it is the first switch in the sequence. If we only touch the "1" piece one time and never make another switch to put it back where it originally was, this can't equal the identity action, so this can never happen!

We just showed that if the identity action can be done in N switches, it can also be done in N-2 switches. This means that the identity action can **NEVER** be done with an **ODD** number of switches, because we could keep decreasing the number of switches by two until we are left with one switch equaling the identity permutation (which is impossible)! So by our contradiction argument earlier, the 15-14 configuration can't be solved in an even number of switches, so it can't be solved by only using moves! **BOOM**!

That was A LOT. The best learning in math doesn't just come from solving a problem, but walking back through it, retracing your flow of logic, and enjoying the beauty of the path you took to get to your solution. Try to think through it yourself, and when you're ready, let's walk through it together:

- We played with the 15 puzzle and got an idea of how to solve it using normal moves

- We started questioning whether the 15-14 configuration and other similar configurations could be solved, and developed a claim
- Our exploration motivated us to define the concept of moves and switches, and explore their structures until we uncovered their differences in parity
- We took an educated leap of faith to generalize our question into a form we could work with
- We simplified the problem by focusing on the identity action and showing that it could only be represented by an even number of switches
- By cleverly contradicting ourselves, we showed that it is impossible for the 15-14 configuration to be solved in an even number of switches
- We realized that since all switches are moves, if we can't solve a configuration in an even number of switches, we can't solve the configuration in an even number of moves.

Notice that the way our "formal" solution flows doesn't follow the flow of our thought process! The journey of discovery is one of the most amazing things in math, and this quote very nicely summarizes how I feel about this:

> "Perhaps I can best describe my experience of doing mathematics in terms of a journey through a dark unexplored mansion. You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture, but gradually you learn where each piece of furniture is. Finally, after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were. Then you move into the next room and spend another six months in the dark. So each of these breakthroughs, while sometimes they're momentary, sometimes over a period of a day or two, they are the culmination of—and couldn't exist without—the many months of stumbling around in the dark that proceed them." - Andrew Wiles

A lot of this problem was abstract algebra in disguise. Through the lens of the 15 puzzle, we explored the concept of **symmetry groups**, systems that allow us to perform actions that rearrange objects. The 15 puzzle can be seen as a list of 16 objects (with the empty space being viewed as "piece 16"). The "switches" that we explored are known as "transpositions" in abstract algebra terminology, and our "shufflings" are known as "permutations." So I could have just opened this paper with:

"Theorem: Any permutation can either be represented by an odd number or an even number of transpositions.

Proof: Consider a string of N transpositions equivalent to the identity permutation…"

But then we miss the fun to be had exploring problems, developing conjectures, jumbling up your thoughts, and feeling the thrill of putting all the pieces together (quite literally).

I hope you enjoyed our journey. Keep playing with the 15 puzzle, there's a lot more questions to be asked and interesting properties to be discovered. What other configurations can't be solved? How many can't be solved, and how many can? How would this puzzle change if there were 24 pieces on a 5x5 grid, or 99 pieces on a 10x10 grid? What if I took my puzzle and bent it into a cylinder, like this, so that the “1” piece was able to switch with the empty space?

Credit: Henry Segerman, Oklahoma State University

Happy puzzling and see you next time!

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