

THE 2021 ROSENTHAL PRIZE
for Innovation and Inspiration in Math Teaching

The Game of Cats
A mathematical logic activity

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Lesson Plan

GRADES

4-8

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1 Overview

The Game of Cats is a mathematical logic activity based on sixteen cat cards and eighteen statement cards. Through a sequence of guided tasks and challenges, students understand the correct usage of the conjunction **and**, of the disjunction **or**, as well as the negation of **and**-statements and **or**-statements. The students learn about the law of excluded middle and about double negation. In an extended class, students explore **if-then**-statements and how statements can be trivially true if the **if** hypothesis is not satisfied. They also explore what it means to negate an **if-then**-statement and how **if-then**-statements and their negations can be written as **and**-statements or **or**-statements.

2 Standards for Mathematical Practice

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Look for and make use of structure.

3 Time Required

Preparation time: 60 minutes

Class time:

Lesson Plan 1 [60 minutes]: Explore and-statements, or-statements, and their negations. Students work on pages 1-4 of the handout.

Lesson Plan 2 [80 minutes]: Explore and-statements, or-statements, if-then-statements, and their negations. Students work on pages 1-5 of the handout.

Additional Discussion [30-35 minutes]: Additional items that can be added to any of the two lesson plans.

4 Materials

Each group of 2-4 students receives one set of cat cards numbered 1-16, one set of statement cards labeled A-R, several make your own cards, and the handout. Printable cards and the handout are attached. It is recommended that the pages of the handout are all printed in different colors, so the teacher can better see that groups are working on the correct task.

5 Lesson Plan

Place students in groups of 2-4 and provide each group with the handout and complete sets of cat cards and statement cards. Start by asking students to briefly look through the cards and then to work on the first page of the handout [15 minutes].

5.1 And-statements, or-statements

- **Round 1:** For which cat cards 1–6 is the statement true?

B “There exist white cats”.

Here clarify that the existence of one white cat suffices and advise students to write down the labels of all matching cat cards on their teams’ worksheet.

Answer: 2, 4, 5, 6

- **Round 2:** For which cat cards 1–6 is the statement true?

D “There exist black cats”.

Answer: 1, 2, 5

- **Round 3:** For which cat cards 1–6 is the statement true?

G “There exist black cats **and** there exist white cats.”

Answer: 2, 5.

- **Round 4:** For which cat cards 1 through 6 is the statement card true?

H “There exist black cats **or** there exist white cats.”

Answer: 1, 2, 4, 5, 6.

Compare your answers in Rounds 1, 2, 3 and 4. Do you notice anything?

The list in Round 3 contains the elements which are on both lists 1 and 2. For statement G to be true, both statements B and D must be true. The list in Round 4 contains the elements which are in either list 1 or list 2, or both. For statement H to be true, at least one of the statements B and D must be true. They could both be true, but that is not necessary.

By completing rounds 3 and 4, students learn the meaning of the conjunction **and** and that of the disjunction **or**. In general, a statement of the form “X **and** Y” is true if and only if statements X and Y are both true. A statement of the form “X **or** Y” is true if and only if at least one of the statements X and Y is true. That is, X is true, Y is true, or both are true.

- **Challenge:** For which days of the week is the following statement true:

“Today is Monday **or** tomorrow is Thursday.”

Answer: Monday and Wednesday

5.2 Negations

Students continue to the second page of the handout [15 minutes]

- **Round 5:** Among all statement cards, find the card which is the **negation** of statement

D “There exist black cats.”

Answer: A “There are no black cats.”

Introduce the notation $\text{not}(X)$ to represent a statement equivalent to the negation of statement X.

- **Round 6:** Among all statement cards, find the card which is the **negation** of statement

E “There exist cats of the same color.”

Answer: F “No two cats have the same color.”

For which cat cards numbered 11-16 is statement E true?

Answer: 11, 13

For which cat cards numbered 11-16 is $\text{not}(E)$, the negation of statement E true?

Answer: 12, 14, 15, 16

Are there any overlaps in the previous two answer? Why is that? Notice that each cat card falls into one and only one category, that is, for each cat card either statement E is true or statement E is false, meaning that its negation, statement F, is true.

In mathematics, *the law of the excluded middle* states that for every statement, either the statement is true or its negation is true (i.e the statement is false), there is no middle ground.

- **Round 7:** Among all statement cards, find the card which is the negation of statement

F “No two cats have the same color.”

Answer: E “There exist cats of the same color.” Here we notice that F is the negation of E and E is the negation of F, that is, the negation of the negation of E. This is an example of *double negation*, $\text{not}(\text{not } E)$ is equivalent to E.

Ask students to give an example of a double negation.

5.3 Negations of and-statements and or-statements

Students continue to the third page of the handout [10 minutes]

The next couple of rounds combine the negation with **and** and **or**. Recall that a statement of the form “X **and** Y” is true if and only if statements X and Y both are true. And a statement of the form “X **or** Y” is true if at least one of the statements X and Y is true.

- **Round 8:** Among all statement cards, find the card which is the negation of statement

K “There exist white cats **and** there exist cats of the same color.”

For each cat card numbered 7–12, which statement is true, K or the negation of K? By the law of excluded middle we know exactly one of them is true.

*Answer: The negation of statement K is statement M “There are no white cats **or** there are no cats of the same color”. For cards 7, 8, 11 statement K is true. For cards 9, 10, 12 statement M is true.*

In general, the statement “X **and** Y” is false if at least one of X and Y is false. They could both be false, but that is not necessary.

- **Round 9:** Among all statement cards, find the card which is the negation of statement

L “There exist white cats **or** there exist cats of the same color.”

For each cat card numbered 11–16, which statement is true, L or the negation of L?

*Answer: The negation of statement L is statement N “There are no white cats **and** there are no cats of the same color”. For cards 11, 12, 13, 14 statement L is true. For cards 15, 16 statement N is true.*

In general, the statement “X **or** Y” is false if X and Y are both false.

5.4 Mystery card challenges

Students continue to the fourth page of the handout [20 minutes]

- **Challenge:** Find the mystery cat card for which all statements are **true**.

- * C “There are more white cats than there are black cats”
- * F “No two cats have the same color”

Answer: 14 One white cat.

- **Challenge:** Find the mystery cat card for which all statements are **false**.

- * O “There are cats which are not black.”
- * F “No two cats have the same color.”

Answer: 1 Four black cats.

This next challenge is slightly more difficult.

- **Challenge:** Find the mystery cat card for which all statements are **false**.

- * D “There exist black cats.”
- * L “There exist white cats **or** there exist cats of the same color ”

Answer: 15 One blue cat.

Time permitting, the next challenge can be played several times. Each team creates a challenge, then challenges are swapped between the teams.

- **Challenge:** Create your own mystery cat card game.

- * Use the blank statement cards to create your own statements or use any statements A-R.
- * The solution to the game should be exactly one cat card 1–16 which the other teams will try to find.

Sample student work is presented in Figure 1.

The lesson up to this point is about 60 minutes. One can stop after the challenges are solved, or can continue with the next activities which explore **if-then**-statements [20 minutes].

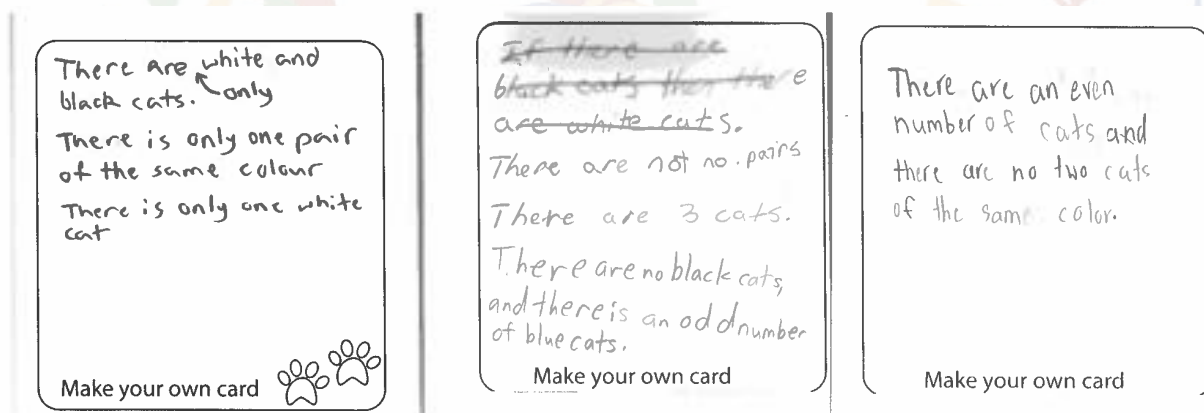


Figure 1: Sample student work

5.5 If-then-statements

In a statement of type “If X **then** Y”, X is called the *hypothesis* and Y is called the *conclusion*. The statement “If X **then** Y” is false when the hypothesis X is true but the conclusion Y is false. In all other instances, the statement “If X **then** Y” is true. In particular, the statement is trivially true if the hypothesis X is false.

- **Round 10:** For which cat cards labeled 1-6 is the statement true?

Q “**If** there exist black cats **then** there are no white cats.”

Answer: 1, 3, 4, 6

Among all statement cards, find an **or**-statement which is equivalent to statement Q.

Answer: Statement I “There are no black cats or there are no white cats”. Notice that if there are no black cats the statement is trivially true, since the hypothesis is not satisfied.

In general, since the statement “**If** X **then** Y” is true when X is false or when both X and Y are true, the statement “**If** X **then** Y” is equivalent to the statement “not(X) **or** Y”.

The statement on which card is equivalent to the negation of statement Q?

Answer: The negation of statement I (and statement Q) is statement G “There exist black cats and there exist white cats”.

In general, the negation of “**If** X **then** Y” is “X **and** not(Y)” (the hypothesis X is true and the conclusion Y is false).

- **Round 11:** Find three cat cards for which the statement true?

R “**If** there exist black cats **then** there exist blue cats.”

Possible answers: 3, 4, 5, 6, 9, 10, 11, 13, 14, 15. All cards with no black cats are included, as well as all cards with both black and blue cats.

Can you write an **or**-statement which is equivalent to statement R?

*Answer: There are no black cats **or** there exist blue cats.*

Can you write a statement which is equivalent to the negation of statement R?

*Answer: There exist black cats **and** there are no blue cats.*

- **Challenge:** Create your own mystery cat card game. Include at least one if-then-statement.
 - * Use the blank statement cards to create your own statements or use any statements A-R.
 - * The solution to the game should be exactly one cat card 1–16 which the other teams will try to find.

6 Additional Discussion

Additional questions [30-35 minutes]

Items 1, 2, 3, and 4 can be discussed at the end of page 1 of the handout. Items 5 and 6 can be discussed at the end of page 2 of the handout. Items 7 and 8 can be discussed at the end of page 3 of the handout.

1. When is the statement “X and Y and Z” true?

Answer: When X, Y, and Z are all true

2. When is the statement “X or Y or Z” true?

Answer: When at least one of X, Y, and Z is true.

3. When is the statement “(X or Y) and Z” true?

Answer: “(X or Y) and Z” is true when Z is true and at least one of X and Y is true.

4. When is the statement “X or (Y and Z)” true?

Answer: “X or (Y and Z)” is true when either X is true or both Y and Z are true.

By comparing items 3 and 4, notice that the placement of the parentheses does matter. The statements in 3 and 4 are not equivalent. For example, if Z is false, then the statement in 3 is false, regardless of X and Y. On the other hand, if X is true, then the statement in 4 is true, regardless of Y and Z.

5. What can be said about the statement “ X **and** $\text{not}(X)$ ”?

Answer: This statement is always false, since X and $\text{not}(X)$ cannot both be true at the same time.

6. What can be said about the statement “ X **or** $\text{not}(X)$ ”?

Answer: This statement is always true, since one of X and $\text{not}(X)$ is true.

7. Explain why the statements “ $\text{not}(X \text{ **and** Y)$ ” and “ $\text{not}(X)$ **or** $\text{not}(Y)$ ” are equivalent.

*Answer: The statement “ $\text{not}(X \text{ **and** Y)$ ” is true when “ $X \text{ **and** } Y$ ” is false. This is when at least one of X and Y is false, that is, when “ $\text{not}(X)$ **or** $\text{not}(Y)$ ” is true.*

8. Explain why the statements “ $\text{not}(X \text{ **or** } Y)$ ” and “ $\text{not}(X)$ **and** $\text{not}(Y)$ ” are equivalent.

*Answer: The statement “ $\text{not}(X \text{ **or** } Y)$ ” is true when “ $X \text{ **or** } Y$ ” is false. This is when both X and Y are false, that is, when “ $\text{not}(X)$ **and** $\text{not}(Y)$ ” is true.*

7 Printable worksheets and cards



Welcome to The Game of Cats!

- **Round 1:** For which cat cards 1-6 is the statement true?

B “There exist white cats”.

Answer:

- **Round 2:** For which cat cards 1-6 is the statement true?

D “There exist black cats”.

Answer:

- **Round 3:** For which cat cards 1-6 is the statement true?

G “There exist black cats **and** there exist white cats.”

Answer:

Compare your lists of labels from Rounds 1, 2 and 3. Do you notice anything about these three lists of labels?

- **Round 4:** For which cat cards 1-6 is the statement card true?

H “There exist black cats **or** there exist white cats.”

Answer:

Compare your lists of labels from Rounds 1, 2, and 4. Do you notice anything about these three lists of labels?

- **Challenge:** For which days of the week is the following statement true:

“Today is Monday **or** tomorrow is Thursday.”



- **Round 5:** Among all statement cards, find the card which is the negation of statement D “There exist black cats.”

Answer:

We denote the statement equivalent to the negation of statement D, $\text{not}(D)$.

- **Round 6:** Among all statement cards, find the card which is the **negation** of statement E “There exist cats of the same color.”

Answer:

For which cat cards numbered 11-16 is statement E true?

Answer:

For which cat cards numbered 11-16 is $\text{not}(E)$, the negation of statement E true?

Answer:

Are there any overlaps in your previous two answer? Why is that?

- **Round 7:** Among all statement cards, find the card which is the negation of statement F “No two cats have the same color.”

Answer:



- **Round 8:** Among all statement cards, find the card which is the negation of statement K “There exist white cats **and** there exist two cats of the same color.”

Answer:

For which cat cards numbered 7-12 is statement K true?

Answer:

For which cat cards numbered 7-12 is not(K), the negation of statement K true?

Answer:

In general, the statement “X **and** Y” is false if at least one of X and Y is false. They could both be false, but that is not necessary.

- **Round 9:** Among all statement cards, find the card which is the negation of statement L “There exist white cats **or** there exist two cats of the same color.”

Answer:

For which cat cards numbered 11-16 is statement L true?

Answer:

For which cat cards numbered 11-16 is not(L), the negation of statement L true?

Answer:

In general, the statement “X **or** Y” is false if both X and Y are false.



- **Challenge:** Find the mystery cat card for which all statements are **true**.
 1. C “There are more white cats than there are black cats”
 2. F “No two cats have the same color”

Answer:

- **Challenge:** Find the mystery cat card for which all statements are **false**.
 1. O “There are cats which are not black.”
 2. F “No two cats have the same color.”

Answer:

- **Challenge:** Find the mystery cat card for which all statements are **false**.
 - * D “There exist black cats.”
 - * L “There exist white cats **or** there exist two cats of the same color ”

Answer:

- **Challenge:** Create your own mystery cat card game
 - * Use the blank statement cards to create your own statements.
 - * The solution to the game should be exactly one cat card 1–16 which the other students will try to find.



- **Round 10:** For which cat cards 1-6 is the statement true?

Q “**If** there exist black cats **then** there are no white cats.”

Answer:

Among all statement cards, find an **or**-statement which is equivalent to statement Q.

Answer:

The statement on which card is equivalent to the negation of statement Q?

Answer:

- **Round 11:** Find three cat cards for which the statement is true.

R “**If** there exist black cats **then** there exist blue cats.”

Answer:

Can you write an **or**-statement which is equivalent to statement R?

Answer:

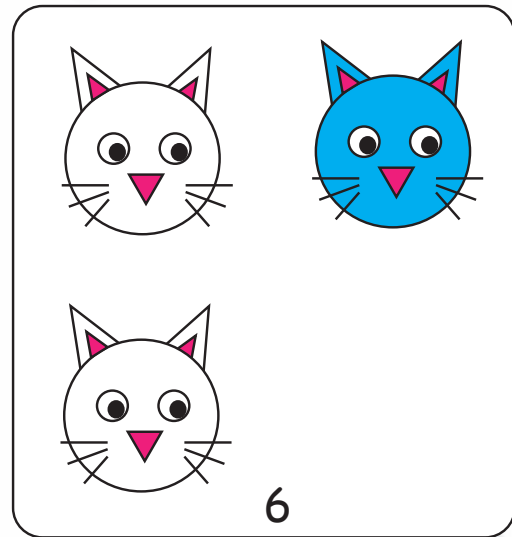
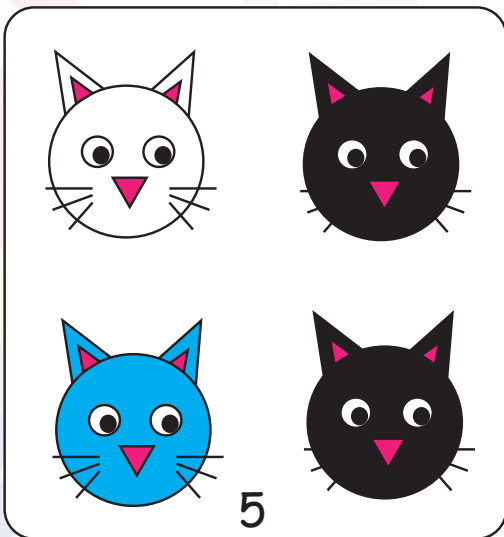
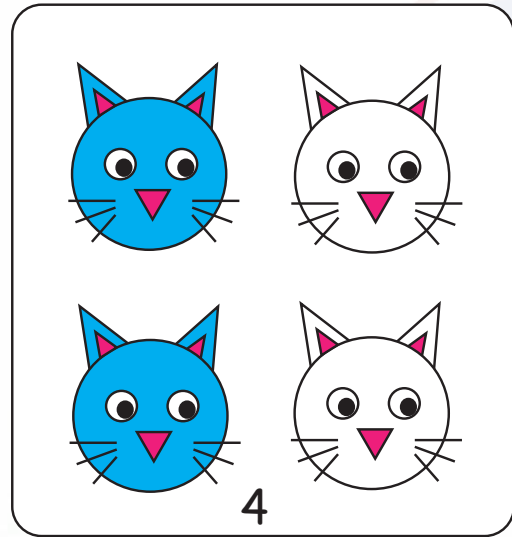
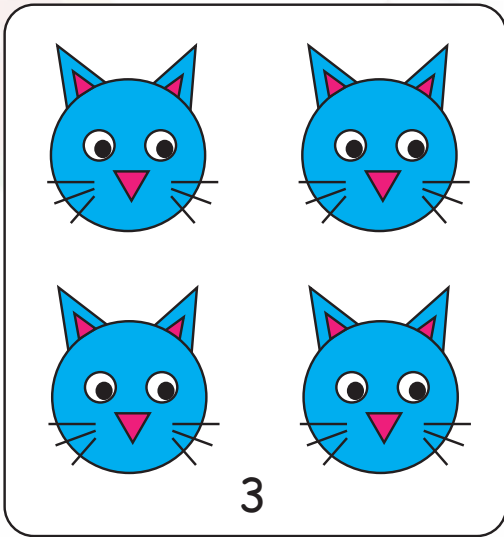
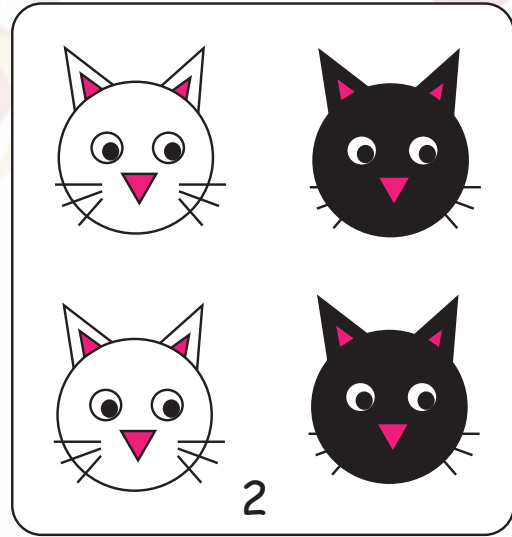
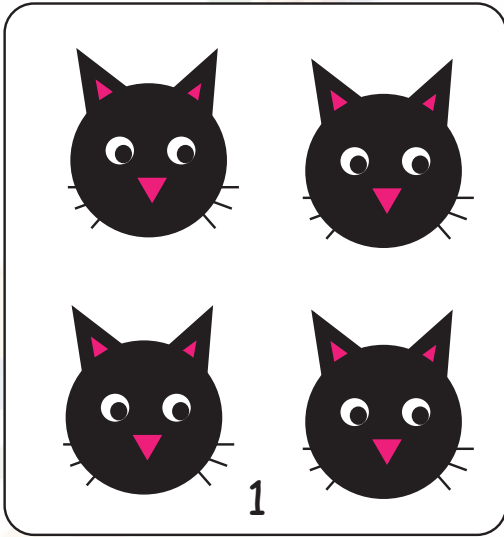
Can you write a statement which is equivalent to the negation of statement R?

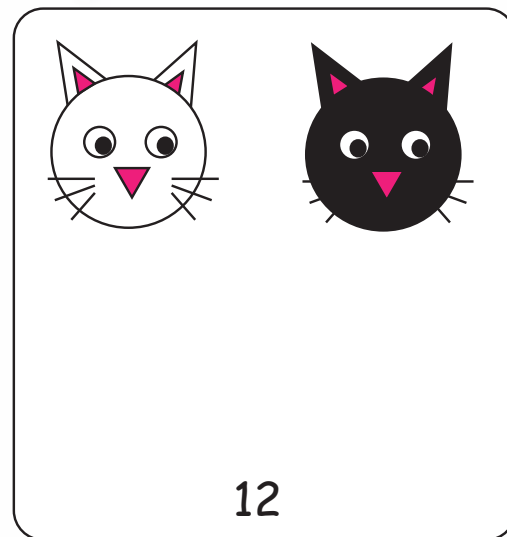
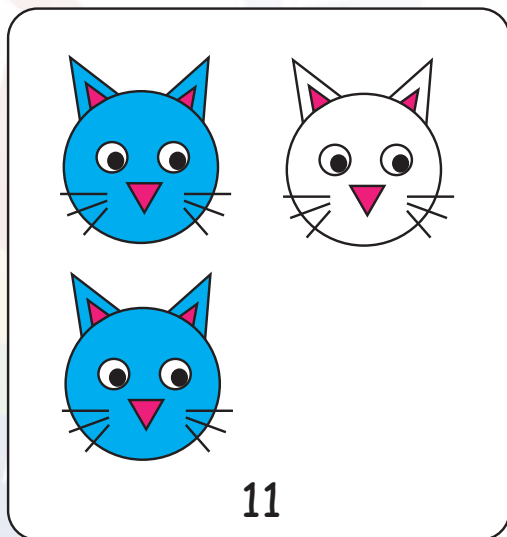
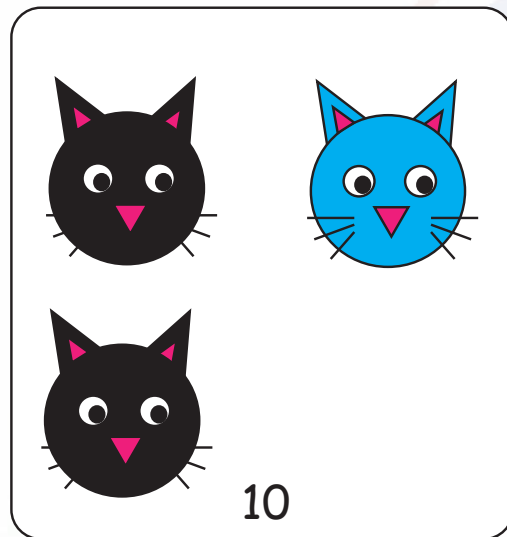
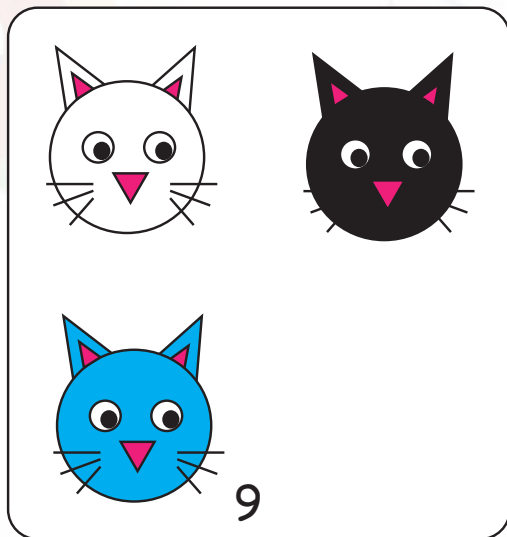
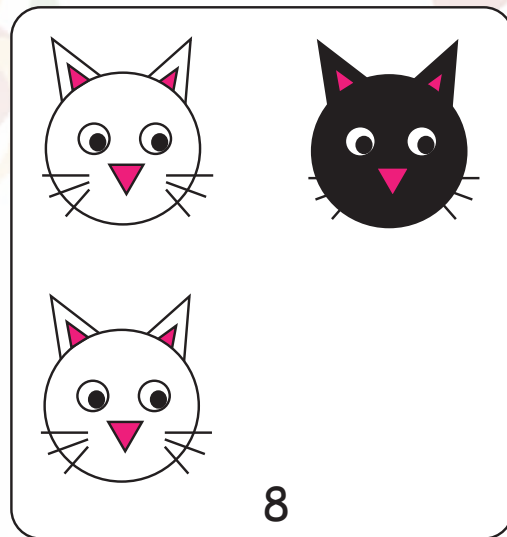
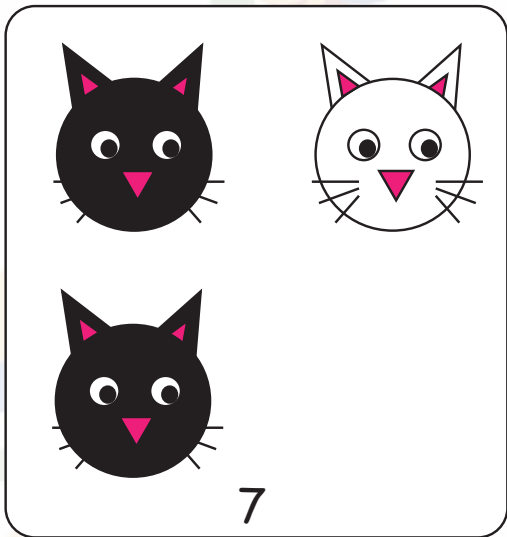
Answer:

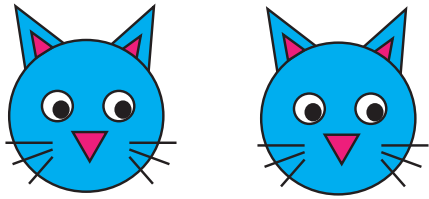
- **Challenge:** Create your own mystery cat card game. Include at least one **if-then**-statement.

- * Use the blank statement cards to create your own statements or use any statements A-R.

- * The solution to the game should be exactly one cat card 1–16 which the other teams will try to find.



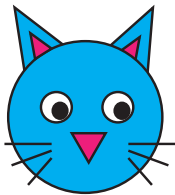




13



14



15



16

Make your own card



Make your own card



There are no
black cats.

A



There exist
white cats.

B



There are
more white
cats than there
are black cats.

C



There exist
black cats.

D



There exist
cats of the
same color.

E



No two cats
have the same
color.

F



There exist
black cats
and
there exist
white cats.

G



There exist
black cats
or
there exist
white cats.

H



There are
no black cats
or
there are no
white cats.

I



There are
no black cats
and
there are no
white cats.

J



There exist
white cats
and
there exist
cats of the
same color.

K



There exist
white cats
or
there exist
cats of the
same color.

L



There are no
white cats
or
there are no
cats of the
same color.

M



There are no
white cats
and
there are no
cats of the
same color.

N



There are cats
which are
not black.

O



There are at
least as many
black cats as
white cats.

P



If there exist
black cats
then
there are no
white cats.


Q





If there exist
black cats
then
there exist
blue cats.


R




Make your own card 

Make your own card 

Make your own card 

Make your own card 

Make your own card 

Make your own card 