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Binary Coins Are Better Than Bitcoin!

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Lesson Plan
GRADE
5

Overview

The objective is to learn about the binary number system in a tangible and interactive way. Students will each be their own business owner, factory, and also mint! They will create their own products and pricing and later on even their own currency! They will first practice with our familiar denominations of coins and experience their weaknesses to make exact change for buying products. Then they will try to improve upon the system by minting their own coins with values of their own choosing. Finally, they will be introduced to a binary coin set which is optimally efficient to span a range of numbers from 1 to $2^n - 1$ with only n coins.

This activity may easily be scaled to conduct it with as little as one student or with an entire class. It is also easily customizable to the instructor's preferences.

Lesson Goals

- Learn about other number systems and particularly the binary number system.
- Learn how to convert decimal numbers to binary.
- Learn that binary is more efficient than decimal in certain real-world contexts, e.g. computers and coins.
- Learn that the number 1—though small—is an essential part of every number system.
- Learn that monetary value does not equal purchasing power. In the context of this activity higher numbers are not able to build smaller numbers, i.e. they grant more monetary value but less purchasing power and thus wealth. So hints of liquidity and inflation are present as a higher nominal number does not always translate to the same purchasing power and may not be able to be exchanged for desired goods. Though liquidity and inflation are not intended as focal points or learning objectives as they are not treated well and are too off-topic for the purpose of this exercise.

Vocabulary

This vocabulary is unnecessary but the reader may find these definitions useful in thinking about the subject matter in the context of this activity and it may help ease the communication with the students.

- number system or numeral system

  A system capable of constructing all the natural numbers. Note that the term "number system" may also refer to more exotic numbers like the complex or quaternion numbers but the term is used herein as appropriate within context. An alternative term would be "numeral system" which is strictly a system that allows you to construct and write the natural numbers.

- base 10 number system

  Our familiar number system utilizing powers of 10 represented by place value with the numerals 0–9.

- decimal

  The base 10 number system accounting for fractional parts. It is often used interchangeably with "base 10 number system" as it is herein but please note that there is a distinction.

- binary

  The number system that uses 2 as a base. Thus powers of 2 are represented by place value and only the numerals 0, and 1 are used. Binary forms the framework of computers.

- money

  A system of value represented numerically that can be traded for real-world products.

- coin

  An instance or package of money with a specified amount.

- denomination

  The amount specified on a particular coin.

- change

  Another word for a coin or group of coins.

- currency

  The collection of all available coins with their respective denominations.

- mint (noun)

  A place where coins are made.

- mint (verb)

  To make a coin.

- monetary value

  The numerical amount of money which is nominal and transient; it may not equate to expected real-world experiences.

- purchasing power

  The ability of a currency to be exchanged for desired goods.

- wealth

  The ability of a person to attain desired goods with the currency they possess.

Background on Number Systems

The decimal system (base 10) is the standard number system used globally. Many languages still count with elements from other number systems, e.g. French and Georgian use vigesimal (base 20) to speak many of their numbers. In English, familiar elements of the duodecimal or dozenal (base 12) number system survive in the words "dozen" for a whole grouping of twelve and "gross" for a grouping of 144. English also has a base twenty word: "score". Some systems of measure are built upon sexagesimal (base 60) such as angles ($360° = 1rev$) and time (60 seconds = 1 minute). Binary (base 2) is used in electronics due to the simplicity and integrity to store and manipulate numbers through the presence of charge or the lack thereof. Hexadecimal (base 16) is used as a fair intermediary that is more human-readable to make sense of the universal binary used at the chip layer. Though ultimately it is the decimal system that is the backbone of all human numeracy and we all use the standard Western Arabic numerals (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) to construct all our numbers shared globally.

Number Systems in Currency

All currencies have been strictly decimalized except for two: the currencies of Mauritania, and Madagascar. Though their coins—based on a quinary (base 5) number system—are no longer in use and arguably their currencies may still appear as decimal externally. However, the coinage in all these decimal currencies is not the fundamental numerals that build the numbers but instead an arbitrary sampling chosen for convenience. For example, in the US we have a 25 cent piece (quarter), four of which make 100 cents or 1 dollar but in the UK the closest they have to our quarter is a 20 pence piece, five of which make 100 pence or 1 pound. If the currency truly mirrored its decimal structure then there would only be denominations for the powers of 10 and when you have ten of one denomination you could replace it with the next one up. Therefore the US does not need nickels and quarters, nor does the UK need 2 pence, 5 pence, and 20 pence coins, etc. Having such coins serves only as a convenience and they are chosen arbitrarily. One may argue that even dimes are unneeded since 10 pennies are equivalent. Indeed the pennies—or in general the value of 1—constitutes its own number system in and of itself: the unary number system. All numbers can be built with enough ones but it is not very efficient, however, the value of 1—which is any number base raised to the zeroth power—is a necessary part of every number system.

Decimal vs Binary

The decimal system uses the ten familiar numerals 0–9 to build all the numbers in conjunction with place values which are built from the powers of 10. So twenty is represented by a two in the tens place and a zero in the ones place or $20 = 2 * 10^1 + 0 * 10^0$. In binary, the place values are instead built from the powers of 2 and only the numbers 0 and 1 are used in each position. Twenty is thus represented by $20 = 1 * 2^4 + 0 * 2^3 + 1 * 2^2 + 0 * 2^1 + 0 * 2^0 = 16 + 4$ and it may be written in binary as the coefficients of the power sequence like in decimal thus $decimal(20) = binary(10100)$. A common notation is to use subscripts to denote the number system being used but it may be more tractable to write it out as shown previously. Since decimal utilizes a higher base number it takes fewer digits to represent numbers but in physical terms, it means more complexity per position and this is why computers are in binary. It is much easier to have two states (high and low or on and off) vs 10 different states per register. The same goes for physical coins; a greater quantity is required of each denomination to build values in decimal than in binary. To build all the numbers from 1–31 in decimal, all ten numerals are needed or 12 coins that represent each place value:

3 x (ten units) and 9 x (ones units). With the denominations in USD, you would need at least 8 coins: 1 x quarter, 2 x dimes, 1 x nickel, and 4 x pennies. Though in binary, only 5 coins are needed to represent all the numbers from 1–31: 1 x (16 coin), 1 x (8 coin), 1 x (4 coin), 1 x (2 coin), and 1 x (1 coin). This is the most efficient in that it requires the fewest number of coins and only one of each denomination to span the entire number range. Some numbers like 31 will require more binary coins than decimal coins or even USD coins but it is the span of the range of numbers that is optimized. In general, n binary coins representing the powers of two $2^0, \ldots, 2^{n-1}$ will span all numbers $1, \ldots, 2^n - 1$.

Setup

1. Calculating the necessary materials per student.

   Each student will need the following:

   (a) One set of coins with the denominations of your local currency. I used USD: 1, 5, 10, 25, and 50. The instructor may like to use more denominations or get into dollars which would work but it is recommended to avoid decimals as they may distract from the purpose. It is also recommended that only five denominations are used as it makes the entire activity more manageable. So although there are six coins under a euro and also six under a pound, the instructor may want to consider using only five denominations for manageability. (5 coins)

   (b) Two sets of blank coins with the same quantity in each set as the local currency set; so five per set in this case. This is variable. The number of blank sets corresponds to how many rounds the students have to try to mint better coins. I only had time to do one round with the class I borrowed but at least two are recommended. (10 coins)

   (c) One set of binary coins again with the same quantity per set as the local currency set. In this case, the denominations are 1, 2, 4, 8, and 16. (5 coins)

   (d) Students may draw their products on regular sheets of paper and cut them out but it is recommended that they be provided with index cards. This eliminates the cutting time, size variation, and it allots their drawing to a reasonable threshold. I found that some students were still drawing masterpieces on the small index cards! Letting them use colors is at the discretion of the instructor. I kept it simple and had them only use pencils which helped save time. Each student should have at least as many index cards as the number of coins in the local currency set so in this case 5. Index cards can easily be bought in many supply places and online for under a cent per card. (5 index cards)

   (e) One copy of the binary coin worksheet at the end of this document. (1 worksheet)

   Therefore it is recommended that each student has 1 worksheet, 5 index cards, and 20 coins: 5 local denominations, 5 binary denominations, and 10 blanks.

2. Minting the coins.

   (a) Punching

   The instructor may decide to have the students cut out their own coins but it is far better controlled and will run smoother and quicker if the coins are provided. Though the coins will all have different values it is simpler to have them all be the same size. This reinforces them as variables able to assume or store any value represented. Otherwise, students may be influenced

to put larger numbers on the bigger coins and this may distract from the exercise. So it is recommended that homogeneous coins are punched out of paper. The diameter is subject to choice but I used 2". A 2" diameter hole punch can be picked up at a local craft store or online for around $10 and papers may be layered to punch multiple at a time. I was able to comfortably punch out 15 of the 2" diameter coins from one standard 8.5" by 11" sheet of paper. Though 20 are possible if you're good! I layered 4 sheets at a time so I was able to punch out 60 coins in less than 30 seconds. Naturally with a smaller diameter more may fit per page and perhaps one could punch faster. At the end of this document printouts are provided as another option. Each page has 20 coins with a 1.75" diameter so students may comfortably cut them out if the instructor chooses that route. Therefore one page per student would cover the recommended minimum. There is one page that covers USD and another that covers other currencies like the Euro and Pound.

(b) Stamping

I simply hand-wrote the numbers onto the coins but another option would be to stamp them with number stamps. You can find them at craft stores or online for under $10 and it would be faster and all the numbers would be uniform. Additionally, you may also use number stickers or simply print out the attachments at the end of this document and either cut or punch them out. One final option would be to just produce all blanks and trust the students to write the numbers the instructor tells them to on the first and last coin set when the time comes.

3. Distributing the money.

It is very important that the students do not see the binary coins until the very end otherwise it will spoil the entire activity. If the instructor chooses to use the handouts and has students cut out some coins to aid in the setup they should separate the binary coins from the handout. When I ran the activity I put each coin set into a separate envelope and labeled the envelope. This kept things very clean and organized and thus very manageable since only one coin set is ever used at a time. Then you just pass out one of each envelope to the students. If the instructor goes with the recommended two rounds of minting coins then that would be 4 envelopes per student. Many envelopes may be purchased cheaply for around a dollar.

Additionally, the binary coin worksheet must be kept and not passed out until the end otherwise the students will see the special numbers that they are trying to work toward themselves.

Directions

Part I: Local Currency

1. Manufacturing the products

Give students a set amount of time to draw products on each index card. I gave them only five minutes but I didn't have much time with them. More time is recommended but it depends on how much time you have budgeted for the activity. It is important in this initial stage to get them excited to make their products. Emphasize that they will be able to sell them in their very own store. They should be thinking creatively about what the products are and artistically about how to represent them. However, it is also satisfactory for students to merely write a word down for a product if they don't feel like expressing their thoughts through drawing. I had some students drawing masterpieces and

some only writing down words. Have them write their names on the back of their drawings once they are done.

2. Pricing the products

   Give students some time to choose prices for their products according to the following two rules:

   - The price must be a number between 1 and 31. Note that in general, if the instructor decided to use n coins then the prices would be between 1 and $2^n - 1$.
   - Each product must have a price that ends with a different digit. For example, a student cannot choose 7 and 17 because both numbers end with the same digit.

   A valid example of a set of prices for their five products is: 7, 12, 21, 25, 30. Note that all numbers in the example satisfy both rules; they are between 1–31 and they all end with a different digit.

   Have them write down their prices for each product below its picture or word.

3. Going shopping

   There are two rules to shopping:

   - Only one item may be bought at a time.
   - Items must be paid for with exact change. There is no overpaying and receiving back the difference nor is overpaying and saying "Keep the change" allowed.

   These rules ensure that the exercise is to practice building numbers additively from the available basis numbers.

   Have students pair up and they will take turns buying and selling to keep things orderly. Each student will lay out their products on display and using only the first coin set—the local currency—the first student will try to buy one item from their partner's marketplace. Both students must verify the math and if it checks out then the corresponding coins are exchanged for the item. Both students must store the exchanged good and money separate from their own inventory and money supply. This is to keep things clean: So that the same product is not being bought back and forth and so that each student can experience the same coin set without it being fortified through collecting more money. After the first student goes the second student then tries to buy something from the first and this is repeated as many times as possible.

   After all the students are done, the instructor should lead the class in a discussion reflecting on the experience. The following questions will help guide the discussion:

   - Was everyone able to buy something?
   - Was anyone able to buy everything?
   - Was someone not able to buy anything?
   - Did anyone spend all their money?
   - What items couldn't you buy and why?

If the USD coin set was being used then item prices ending with 2, 3, 4, 7, 8, and 9 could not be purchased; try to collect this information from them. The instructor may write these numbers on the board to emphasize the weaknesses of the coin set. Or the instructor may explicitly write all 18 numbers that cannot be built from the coins on the board: 2, 12, 22; 3, 13, 23; 4, 14, 24; 7, 17, 27; 8, 18, 28; 9, 19, 29. Illuminate the fact that there is—certainly in most cases—money left over that is unable to purchase any remaining products; this is our first experience that purchasing power and monetary value are not always equal. Expound that the coin sets had a total of 91 cents—if you're using USD denominations with the recommended five coins—and yet it still struggled to purchase items that were 31 cents or below. Students are likely to remark that they need more coins; just with more copies of the same coins, they would be able to buy what they couldn't before. The instructor may respond affirming that that is correct however unfortunately in our scenario we only have these five coins available.

Part II: Personal Currency

1. Minting coins

   Return the products to their creators and the coins to their original owners. The marketplaces need to be restored to their original inventory. They are now done with the first coin set and can set it aside. Recall that their names are written on the backs of the products but that shouldn't come into play yet as they are in pairs; so it should be easy to return everything.

   Then building from the conclusion of the prior part, the instructor should ask if the students think they can make better coins that are more able to buy things. Tell them they are now going to mint their own coins! They are now mints that are making their own money! Emphasize that they should take what they learned from the first round to try to make their coins better so that they may buy more things. They will take the first blank set and have to choose five numbers and mark a number on each coin. There is only one rule to making the coins: the value of each coin must be a number between 1–50. Therefore if they want to make five copies of the same coin because they think that might be best then that is ok. Also, have them write their names on the back of the coins because it is their currency. If there is enough time the instructor may also let them decorate their coins with a logo or seal on the back at the instructor's discretion.

2. Going shopping

   Once again students take turns buying one item at a time from their partner's store until they cannot buy things anymore. They still must pay in exact change in order to be able to buy a product. Note that anyone who chose numbers above 31 will certainly not be able to buy anything as the most expensive price possible is 31. After they are done the instructor will lead them in discussion again by asking if their currencies were better than the first set. Instructors may also follow up with the same or similar questions from the end of round one:

   - Was everyone able to buy something?
   - Was anyone able to buy everything?
   - Was someone not able to buy anything?
   - Did anyone spend all their money?
   - What items couldn't you buy and why?

The instructor should also look out for some students who chose large numbers and had more monetary value but less purchasing power. This should be illuminated as an example.

The other thing to look out for is students are likely to look at the five products in their partner's inventory and merely copy the prices onto each of their five coins. Clearly, this is a perfect strategy to buy out their entire stock but it is a trivial solution that only works precisely with only those numbers. Draw attention to this because then we are going to switch things up.

3. Shopping elsewhere

   Have students return all the products and coins to their creators. Now have them switch partners and go shopping again! In most cases, they will find that their personal currency is less effective at buying products in the new marketplace. After this round ask them if their currency performed better or worse in the new market. Now the instructor may truly expound that the perfect strategy of matching the coins to the products is not very good in general. As usual, the instructor may also ask the same probing questions:

   - Was everyone able to buy something?
   - Was anyone able to buy everything?
   - Was someone not able to buy anything?
   - Did anyone spend all their money?
   - What items couldn't you buy and why?

   They will all have different coin sets but the instructor should still try to collect some numbers that could not be built to share with the class. Likewise, currencies with numbers that worked well should be shared with the class and it should be discussed why they did well.

4. Back to the mint

   Have the students return all the products and coins again to their creators. Have them set the coins aside as we are now done with that coin set. Building once again from the prior experience, ask the students to try to create a better coin set again that is able to buy more things and to buy things that it couldn't before. This time the instructor should explicitly point out that there are only 31 prices available amongst the products. The instructor should challenge the students asking if it is possible to come up with five numbers good enough to be able to make any of the 31 numbers. Then have them mint their coins again remembering to also label them on the back.

5. Shopping again

   Then have them go shopping again with their last partner observing all the usual rules. This time though the instructor may allow them to switch partners without replacement of the items to see if they can spend all their currency. After that, the instructor may have them return everything and do another one or two rounds. The instructor then concludes the session with the usual discussion and asks the usual questions.

   By this point, the class may have enough experience to contribute some particular numbers to the discussion that might be important. For instance, the number 1 is necessary to build any two consecutive numbers. Therefore it must be one of the numbers. The instructor may explicitly guide this thinking if it is not volunteered by asking questions like:

- Did anyone use the number 1?
- Do you think the number 1 is an important number to include?

It's also worth stating that if there were enough copies of 1 any number could be built from that coin system. So if someone chose to mint five copies of 1 for their currency then they could build all the numbers from 1–5. That is much less than the 1–31 range but it is without any holes or gaps and can provide a clue. This is worth talking about. The goal of this discussion is to also see whether the class can collectively make a better coin set. Another clue could be to select someone's currency at random and add up all their coins for the total. It is likely to not be 31. If it is smaller then the coins clearly could not even afford the most expensive item. If the total is larger than 31—like 45 for example—then the instructor could point out that there is no need to even be able to make 45 as nothing costs that. So it is inefficient—unnecessary. This should further push students to be thinking not just about the individual numbers but all the combinations of them.

It is likely that after this discussion they may want one more chance to make their own currency and the instructor can let them if there is enough time and materials available. If the recommendation is being followed that means an additional 5 blank coins per student. If the instructor decides to allow this they may conclude it with the usual discussion and then proceed to the next and final part.

Part III: Binary Currency

1. Shopping Spree

   Have the students return all the products and coins to their creators again and store the coins aside as we are finished with them. Finally, they may open the last envelop containing the binary coins and proceed to go shopping with their last partner continuing to observe the shopping rules. The instructor can then the students return the coins and items and have them repeat doing several rounds of shopping with different partners each time. Then finally the instructor may allow them a final round with switching partners without replacement to see if they can spend all their currency.

   After this the usual discussion follows where the same questions may be asked:

   - Was everyone able to buy something?
   - Was anyone able to buy everything?
   - Was someone not able to buy anything?
   - Did anyone spend all their money?
   - What items couldn't you buy and why?

   Students should realize that no matter the price, they were able to buy any product from any market. They could not buy all the products cumulatively but they could buy any one product regardless of price. They could also likely always spend all their money in the final round where they switch partners without replacement.

   The instructor should inquire why the numbers were so versatile. They included the number 1 which should have been established as very important in a prior discussion. The sum of the five coins is also 31 which is the maximum of the range and thus it doesn't jump include any outlier numbers. However, these are not the only five numbers that would sum to 31. Recall that 1 was necessary to reach consecutive numbers. Continuing this logic 2 would be necessary to reach numbers that are 2

away from each other. Then it follows that 3 would be necessary to reach a number 3 away from another however 3 is unnecessary as a coin because they have 1 and 2 which make 3. So 3 may be skipped. Then to build numbers 4 away from each other 4 is needed and that's the next coin. This is an inductive way to explain where the numbers come from. The next number 5 would not be necessary as its own coin because it may be formed from the 1 and 4. Then 6 is covered by 2 and 4. Seven is covered by 1, 2, and 4. So the next number is 8 and so on. The instructor should walk through this procedure with the class all the way to 16, the final coin if the recommendation is being followed.

2. Writing binary

   The instructor can then talk to the students about how these numbers are all powers of 2 and form what is called the binary number system. A number system that works just like our decimal number system but with powers of 2 instead of powers of 10. Since it is so efficient in spanning the numbers in a range it does not require multiple copies of any particular number or coin to build the numbers. So the number 5 would require 5 pennies, that is 5 copies of the same coin whereas it would require one binary 4-coin and one binary 1-coin which are different coins. Binary always has one copy of a number or it doesn't have it at all. So the other difference in the binary number system is that only the numbers 0 and 1 are needed to build and represent any number. This is why computers use binary to store and manipulate numbers internally because it is simple and efficient! Then it is translated for us on the screen so we can see regular decimal numbers.

   The instructor can now pass out the binary coin worksheet at the end of this document. It should be simple enough to understand. The decimal numbers 1–31 are on the left and we are just writing down which binary coins we need to build them, effectively what they were already doing. Then by taking note of which binary coins are used they can easily translate the decimal numbers into binary! Coins that are used get a 1 in that place and coins that are not used get a 0. Then 0's at the left of the number can be dropped just like in the decimal system where 005 is just 5. The worksheet already has the number 5 filled out as an example. The instructor may decide to lead the class together in filling it out collaboratively if they want to.

   After the worksheet is complete the students may notice some patterns since the numbers are all listed in order. The instructor should ask what patterns the students see and discuss them. Now, these well-seasoned students are manufacturers, business owners, minters, and they can all convert decimal numbers to binary!

1	5	10	25	50
1	2	4	8	16

1	2	5	10	20
1	2	4	8	16

Decimal Number	16	8	4	2	1	Binary Number
1						
2						
3						
4						
5			1		1	101
6						
7						
8						
9						
10						
11						
12						
13						
14						
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