MOMATH
NATIONAL MUSEUM OF MATHEMATICS

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TOOTI TOOTI (2222)
create a tessellation from an ordinary envelope!

Chaim Goodman-Strauss

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Classroom materials are linked from this symbol

What Tooti Tooti is

Make a tessellation by cutting open an envelope!

In these activities, we will use an envelope to create planar patterns with four kinds of 2-fold rotational symmetries, and we will explore a connection between these symmetries and the four corners of the envelope: we call this type of symmetry 2222 (Tooti Tooti!)

This lesson is just one example of a wider mathematical phenomenon: *every* planar symmetry has a kind of "envelope" associated with it, and we can learn mathematical facts about symmetry by studying these.

Rotations and Symmetry

Here are the important ideas that we'll need:

Transformations are motions of the entire plane. There are different kinds of transformations. For example a "translation" shifts everything over.

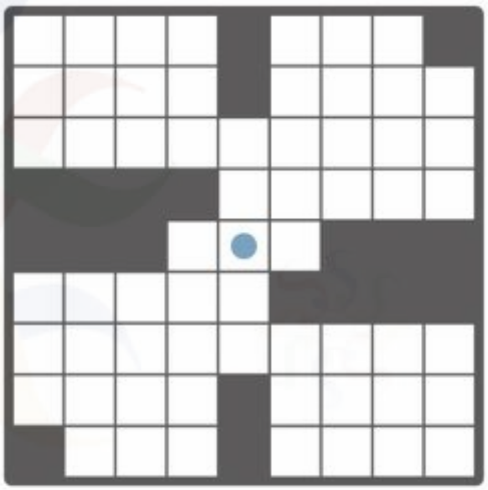
Rotations pivot the plane around a point, a **rotation point**. In this activity we are especially interested in **2-fold** rotations,
rotations by 180 degrees, halfway around a circle.

A **Symmetry** of any kind of pattern is a transformation that leaves the pattern the same.
For example, here are two photos of the same pattern.
What do you think: *Did I rotate the pattern between the time I took the first photo and the time I took the second photo?*

Maybe, maybe not. You just can't tell! The pattern has a 2-fold rotation as a symmetry.

Here are more patterns: check that they have 2-fold rotational symmetry.

Print these out, put a pencil down on the marked 2-fold rotation points — and give these a spin!



The letter **N** has a 2-fold rotational point. The letter **H** can be rotated as well, but an H may also be mirrored vertically and mirrored horizontally. H has a different symmetry type than N does.

Which letters of the alphabet have only 2-fold rotational symmetry, like N?
Which have the same type of symmetry as H?

Did you find them all?[1]

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[1] The letters O, and I and X have the same kind of "kaleidoscopic" symmetry as H does.
The letters N S and Z have just 2-fold rotational symmetry. What about the other letters? ABCD and E all have one mirror symmetry. Some letters, like F and G have no symmetry (besides the symmetry of doing nothing at all!) *Which letters are which?*

Infinite patterns can have more than one 2-fold rotation point.

In this infinite line of b's and q's, there are infinitely many 2-fold rotation points— check that there are 2-fold rotation points between each b and its neighboring q's.

Print out and assemble your own line of b's and q's and try it out!

bqbqbqbqbqbqbqbqbqbqbqbq

But there are only two *kinds* of 2-fold rotation points in this pattern. *Let's take a closer look:*

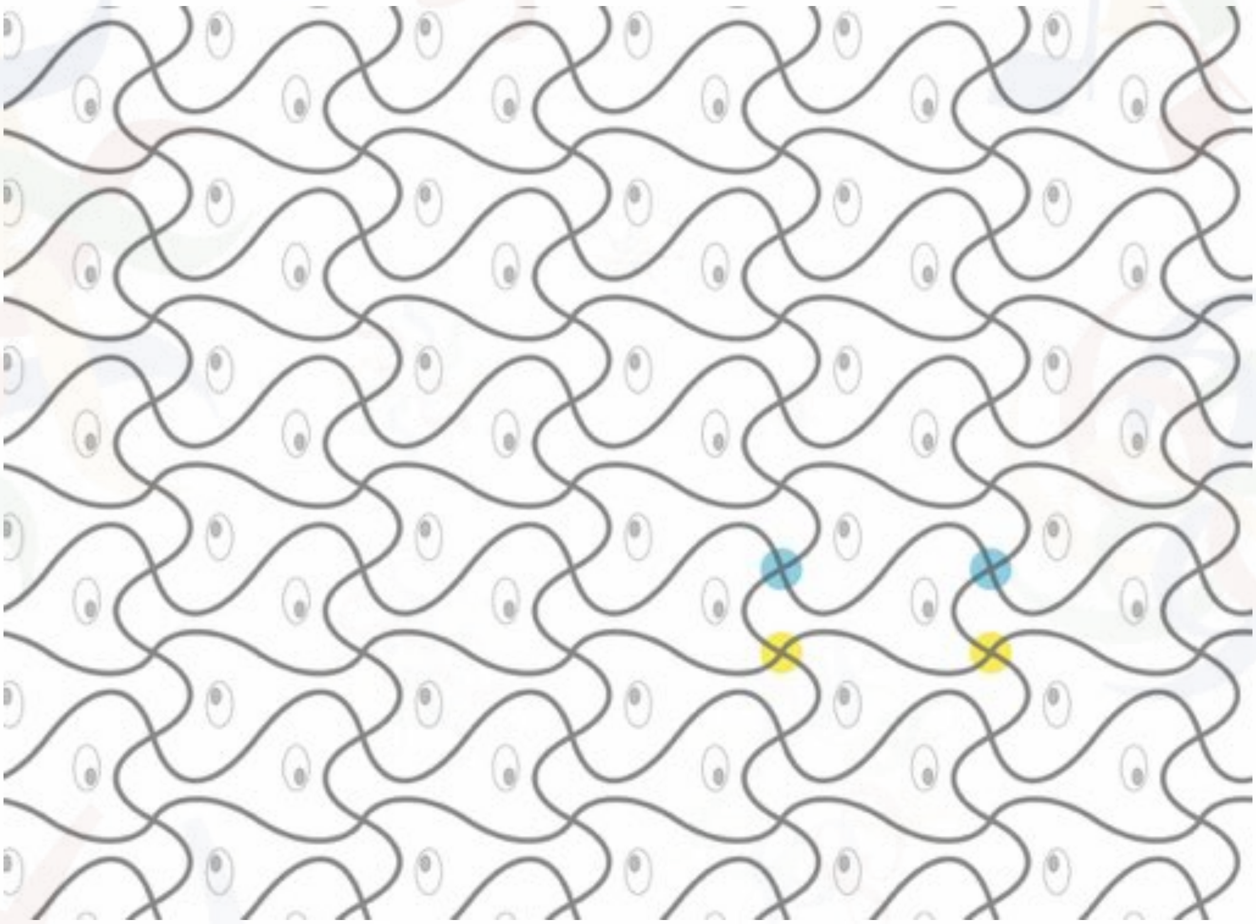
The 2-fold rotation points between the round parts of b and q are all the same as each other in the pattern.

The 2-fold rotation points between the straight backs of q and b are all the same as each other too, and there are no other kinds.

Here's a fun pattern to analyze! Print it out and look for the 2-fold rotation points!

A **tessellation** is any pattern made of tiles, covering the entire infinite plane, like this one (extended forever in each direction).

Print out a few copies of the linked file and put them together to make a larger pattern.



This tessellation has many symmetries, transformations that leave the tessellation unchanged.

A few 2-fold rotation points are marked. The two blue 2-fold rotation points are the same kind: at the top of two fishes' mouths (turn the picture around to check!)
Where are other 2-fold rotation points of this kind?

The yellow points are a different kind of 2-fold rotation point: they are at the bottom of the mouths. *Where are the other 2-fold rotation points of this kind?*

There are a couple of other kinds of 2-fold rotation points — can you find them?

These aren't that easy to spot! The answer is here, and in the Alternate Activity, we'll show a quick way to find all of the kinds of rotation points in this pattern!

Getting Ready

Outline of the Activities

These materials are flexible, and the activities may be used individually or together in any combination. These are the suggested outlines:

For a twenty-minute activity in any group setting:
  * the alternate activity.

For a one-hour middle-school classroom activity:
  * a quick warm up,
  * the first activity,
  * discussion,
  * the second activity.

For a one-hour elementary school classroom activity:
  * a quick warm up,
  * the first activity,
  * discussion,
  * the alternate activity.

For a longer series of activities at any level:
  * spend more time with the warm up materials,
  * the first activity,
  * discussion,
  * the alternate activity,
  * the second activity,
  * and then further explorations.

Organizing the Classroom

Ideally students will be working around flat tables; for the second activity, the students will need to be able to trace their tiles, on butcher paper, a whiteboard table, with sidewalk chalk, etc.

The activities are written for a teacher who is actively circling the room, stopping at each table, asking and soliciting questions, checking with each student, smoothing out technical issues. Younger students may need more assistance with managing strips of tape. Additional whiteboard space, whiteboard magnets, and a document camera may also be useful.

Student Materials

For the first activity, students will need

* a copy of the page linked at right, printed on office paper
(make sure that it is printed at "actual size", not shrunk to "fit media").
* scissors,
* tape,
* pens or pencils; preferably black, red and blue.
It is fine to substitute other colors, but make a consistent choice for the class.

For the alternate activity, students will need the second page linked at right, printed on office paper, and tape and scissors.

For the final activity, students will need
* a pen or pencil,
* scissors,
* a blank notecard envelope, or a piece of office paper and tape,
* a way to trace their tessellations, and space to do this.

A Warm Up Lesson

Before using Tooti-Tooti, students should be familiar with the idea of rotation and symmetry, and of 2-fold rotation points in a symmetrical pattern.

Using the printed materials linked at right, suitable portions of the introduction can be used as a warm-up lesson, showing rotational symmetries by rotating the patterns under a document camera.

(Tip: When demonstrating a rotation, put a pencil tip at the rotation point, to make it clear where it is.)

Strips of b's and q's can be joined together, and a larger sheet of the fish design can be assembled, to suggest an infinite pattern.

Tooti Tooti

The First Activity (20-25 minutes)

Each student receives a printed design, which they tape together into an envelope. After some preparation, each student cuts open their envelope to discover that:

They each have a **tile;**

These tiles all fit together into a pattern with a particular **symmetry;**

Each tile can be rotated to its neighbor around **2-fold rotation points;**

There are **four different kinds** of 2-fold rotations points, and these rotation points are **exactly at the corners of the original envelope!**

Prepping the envelopes (10-15 minutes)

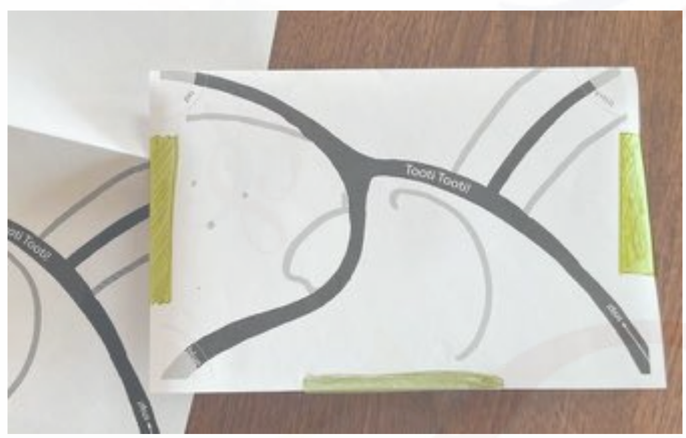
Pass out the scissors, tape, pens or pencils.

Pass out one copy of the printed sheet.

Ask the students to fold the paper over,

and then tape the sides to form an envelope shape.

You don't need much tape: just enough to hold the envelope together. Don't worry about taping all the way to the corners!



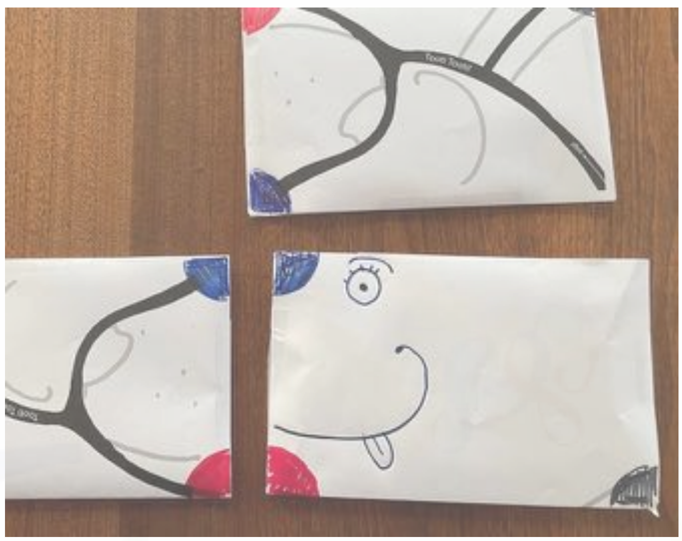
Next, the students will have noticed that the corners of the envelope have “red”, “black”, “blue” and “snip” printed upon them.

(But some of this may have been cut off by the margins of the printer!)

Color in the corners — front and back,
and all the way to the edge of the paper.

Now let’s take a closer look at the decorations on the envelope. Students will see all sorts of unintended things but someone will comment on what seems to be a dog’s smile.

Students will not fully understand this next step, but do not spoil the final effect and overexplain this to them (And to maintain some mystery, *do not* show them this picture!)



The smile is shorter than any dog’s mouth. Imagine that the dog’s snout wraps around to the back of the envelope:

Where does the smile continue? Draw the smile wrapping around onto the back of the envelope.
(It’s fine if it ends up being in a wacky place.)

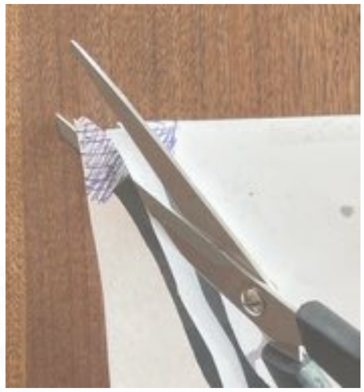
There is an ear printed on the envelope (at least something that plausibly looks like an ear).
A dog’s face is below its ear, and also can continue onto the back of the envelope.

Where do you suppose an eye might be? Draw an eye where you think it might belong.

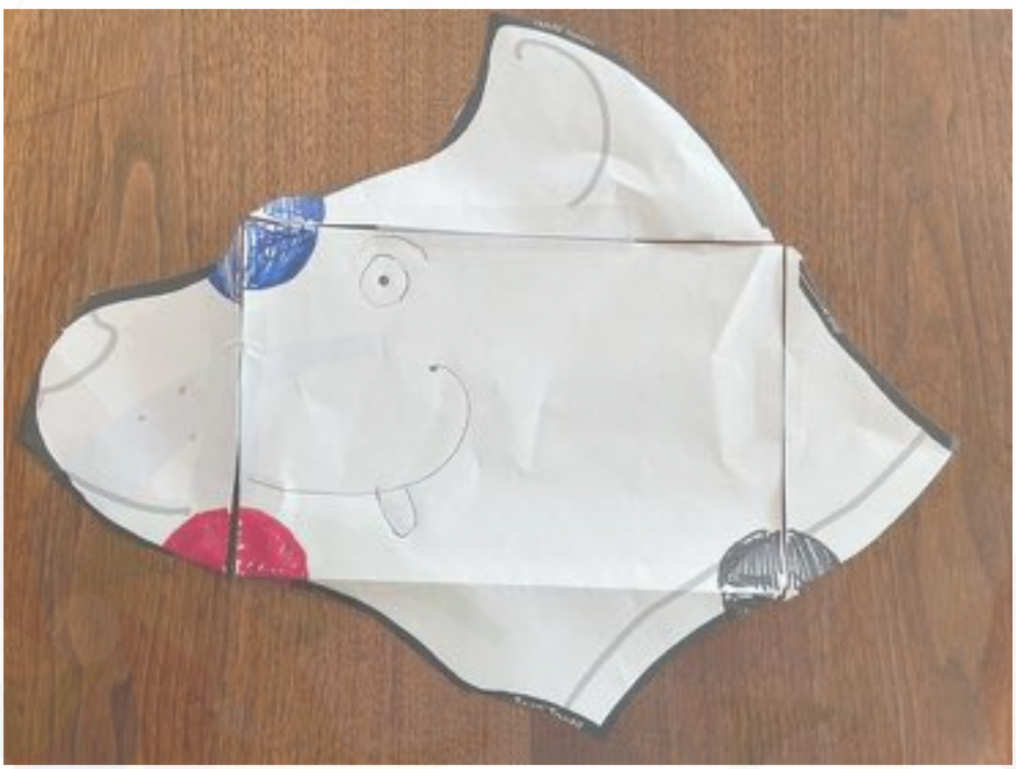
The Effect (10-15 minutes)

Put one end of the scissors in the hole at the corner labeled “snip,” so that you can only cut through the paper on the printed side of the envelope.

Cut along the thick black lines printed on the envelope, only on one side, all the way to each corner.



And the moment of great anticipation. Unfold the envelope, and presto!

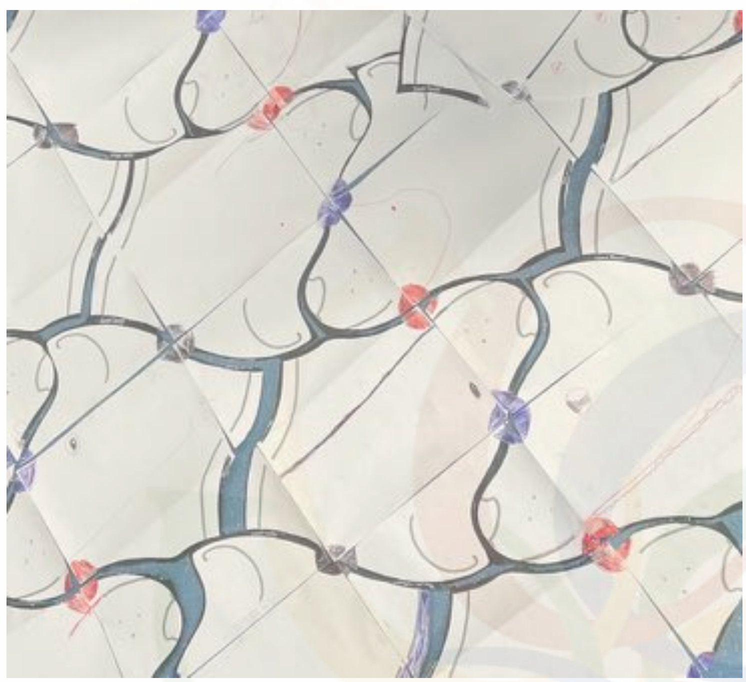


As students complete cutting open their tiles, ask

How do the tiles fit together? How can the tiles form a tessellation?

Once each student has completed a tile and fitted it together into a larger assembly, gather the class's attention.

Discussion (5-10 minutes)



For class discussion prompt:

- *What is a rule for how the tiles fit together?*
- *What transformations take one tile to its neighbor?*
- *How many different kinds of 2-fold rotation point are there in the pattern?*

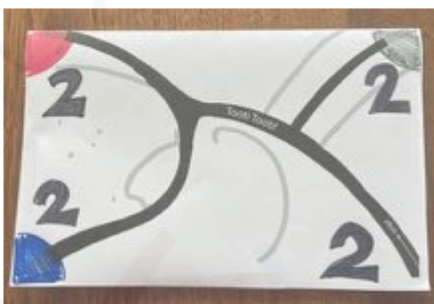
Solicit and amplify answers such as

The tiles fit together so that corners of the same color match.

The transformations are 180 degree rotations — 2-fold rotations — and the 2-fold rotation points are at the colored corners!

All of the rotations about any red corner are the same kind as each other. All of the rotations about any blue corner are the same kind as each other; the rotations around the black corners are all the same; and the rotations around the snipped corners are all the same too.

These correspond, of course, to the four colored corners of the envelope.

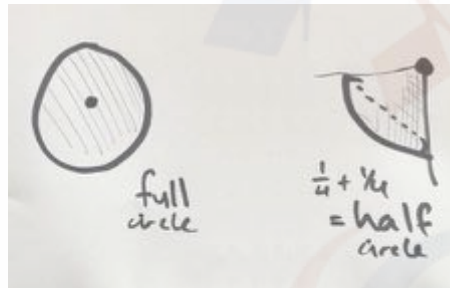


Emphasize to the class: ***We call the symmetry type 2222, Tooti Tooti, the four 2s corresponding to the different kinds of rotations in the symmetrical pattern and to the different corners of the envelope that produced it.***

Corners and rotations

But *why do the 2-fold rotation points exactly correspond to the 2-fold corners of the envelope?*

Each point in the plane is surrounded by a full circle's worth of stuff. On most places on an envelope the same is true. Most points are surrounded by a full circle of paper.[2]



But the corners of an envelope are surrounded by only *half* a circle of paper: one quarter on the front, plus one quarter on the back.

To demonstrate this to the class, cut off a corner of an envelope.

Notice that this shape really is a cone — *round it into one*. This corner is called a **2-fold cone point** because it takes two of them to have full circle of paper.

What happens when we cut to the cone point and unfold?

Cut open your cone all the way to the 2-fold cone point at its corner.

The shape you have when the cone is laid flat must be able to fit together with a copy of itself!

Show the class this by tracing out the shape and fitting it with its copy.

For example cutting open the cone at left produces the shape at right, which can fit together with a copy of itself by a 2-fold rotation.

[2] Even points on the edges of an envelope, if anyone asks. There is a half circle of paper on the front and a half circle of paper on the back, for a full circle total.

Cone points correspond to rotation points.

In Tooti Tooti, 2222, the four 2-fold corners of an envelope correspond to the four kinds of 2-fold rotation point in the tessellation.

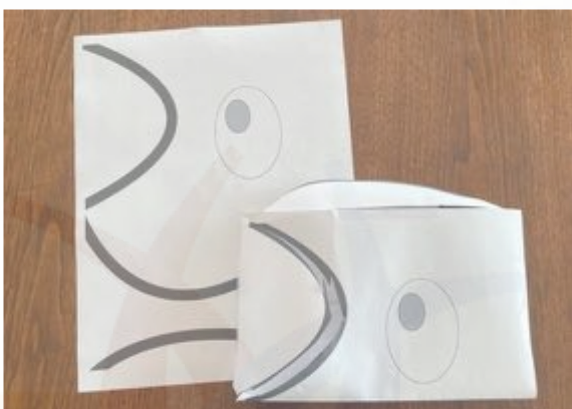
An Alternate Activity

This activity works well on its own or can be used to reinforce the first activity on the way to the second one. For younger students, this activity may be a good alternative to the second activity.

The students will anticipate your instructions to fold and tape up the printed paper into an envelope, then cut it apart and fit it together with their classmates' tiles.

Coloring the corners is optional, but will reinforce the lesson that the corners of the envelope correspond to the different kinds of 2-fold rotation points in the pattern.

Notice that the cutting pattern is printed on both sides of the envelope — warn the students that they should not cut all of the way through!



In the Warm Up exercises, we asked: *Where are the different 2-fold rotation points in this pattern?* Now ask: *Can we find them now?* (They are at the four different kinds of corners of the printed envelope.

The Second Activity (20-25 minutes)

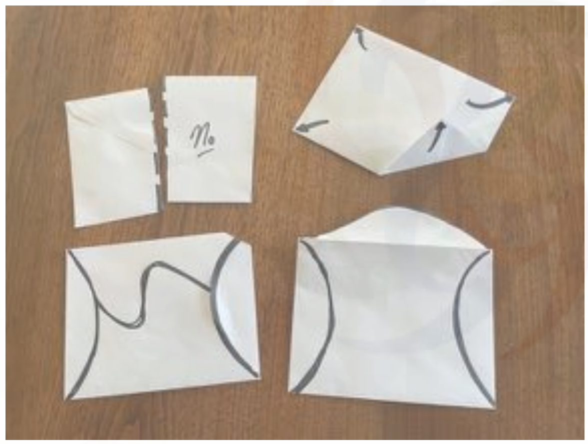
For the second activity, each student receives a blank envelope or a piece of office paper.

If students are using office paper, fold the paper over, and tape the sides together to form an envelope.

If the students are using actual envelopes, be sure that the students seal the flap.

Explain the plan of the activity: the students will draw a cutting pattern, then cut open the envelope and form a tessellation with 2222 symmetry, a tessellation with four kinds of 2-fold rotation point.

What properties should a cutting pattern have?



Solicit or point out that the cuts:

• *must not cut the envelope into more than one piece!*

• *must reach every corner;*

• *must all be connected up to each other*;

• *may cross over to the other side of the envelope if you want to try this out;*

• *and really shouldn't be crazily intricate* — you have to cut these out!

Next, *draw a cutting pattern on your envelope and check that the conditions are met:* (*Do your cuts cut the envelope into more than one piece? Do they reach every corner? Are they connected up?)*

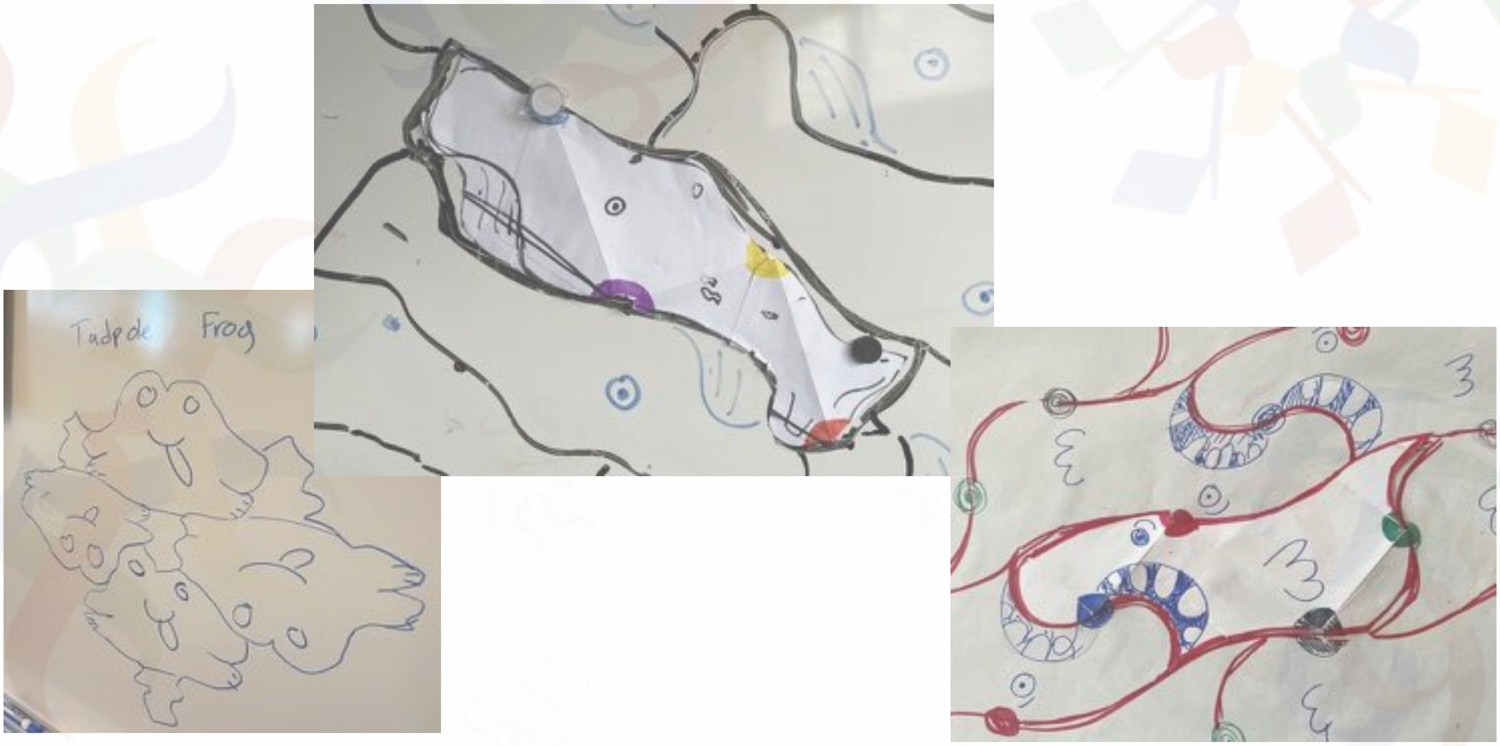
If students are using a prepared envelope, they will need to cut off a corner to be able to get the cutting started.

Warn the students that they should not cut through both sides of the envelope.
As they cut, some students will realize they need some help:

> If the envelope falls into more than one piece – they need fewer cuts!
> If the envelope does not open flat, check they have cut every corner and that all their cuts are connected up to one another.

Each student then cuts open their envelope and imagines what the unfolded shape might be.

They each trace out the symmetrical pattern on a whiteboard or butcher paper, on the sidewalk with chalk, etc.



(Tracing Tip: Mark the corner/rotation points as you go, to help line up how the tiles fit together. Remember that each corner point is a 180° rotation, and so you can simply pivot your tile around such a corner to get the next tile over to trace.)

The Big Idea

An envelope is a kind of mathematical surface.

The main point, and it is a big one is

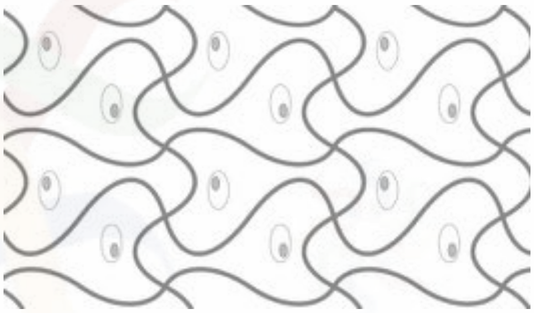
“Symmetry types = surface types”

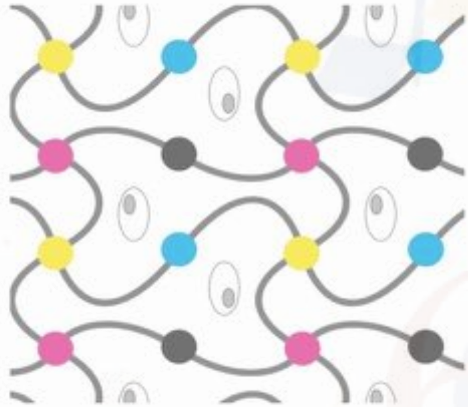
The 2222 **symmetry type** corresponds to the 2222 **“orbifold”**, which an envelope happens to be. This relationship is called the Conway Orbifold Theory of Symmetry.

For Further Investigation

Find rotation points in 2222 symmetry patterns

Here are the rotation points for the fish tessellation in the introduction. Did you find them all?





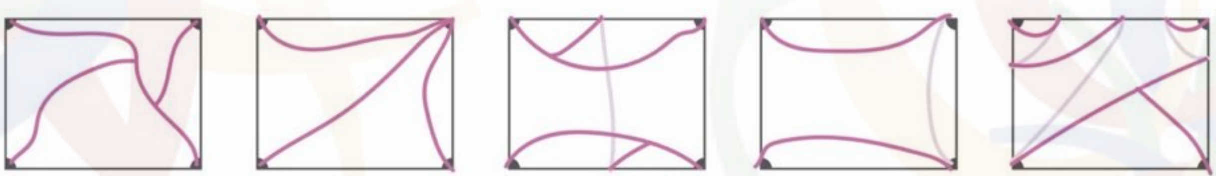
It's not so easy to see them at first, but [if we have the orbifold already](#) it's a snap!

Here are other designs to print out. (Fit copies together to form larger pieces!)

For each pattern, *where are the rotation points?*

Find new ways to cut patterns on an envelope

Here are some more complicated cutting patterns. What other cutting patterns can you come up with?



Explore different shapes of 2222 symmetry pattern

The 2-fold rotation points in a 2222 pattern lie on a grid.

This grid *could* be a square grid. What shaped envelope would that use?

Here are two different rectangular grids. What shaped envelope would they use?




Any lattice of parallelograms can be the scaffolding for a 2222 symmetry pattern!   Here is one example, but try your own.


Let's print out these tiles and find out. For each tile
What 2222 pattern do copies of the tile form? (Trace or assemble a tiling with copies of each tile to find out!)

What 2222 orbifold was this cut from? (Reattach the sides of each tile to find out!)


(The next question is a bit of a spoiler.)

- Show that four copies of any acute triangle can form a tetrahedron, and each corner of this tetrahedron is a 2-fold corner.

Four copies of a right triangle form an envelope, which is just a very flat tetrahedron!!

What about four copies of an obtuse triangle?

Further Still

Tooti Tooti, 2222, is one of just 17 infinite planar symmetry types, in the Euclidean plane, each with its own associated orbifold, as described John H. Conway's *The Symmetries of Things*.

And that's just the Euclidean case — the orbifold notation extends seamlessly to hyperbolic and spherical geometry as well!

Conway's Orbifold Theory of Symmetry

Why settle for anything less! The orbifold theory is a modern and complete approach to planar symmetry. The orbifold notation, such as 2222 in this activity, describes symmetrical patterns in a natural way.

The orbifold notation names are well-defined. They are fully systematic and smoothly generalize. The symbols themselves can be calculated with and manipulated in meaningful ways.

Conway's Magic Theorem proves there are 17 planar symmetry types, by a simple arithmetic count: Each part of a symbol costs some amount and symbols for planar symmetries cost exactly $2, the same 2 as in Euler's Map Theorem! The spherical symmetry types amount to less than $2 and the hyperbolic ones to more than $2.

The orbifold theory is approachable: This activity demonstrates the correspondence between cone points on an orbifold — the four corners of an envelope— and rotations in the pattern.

Overall, the *topology* of the orbifold captures *symmetries* in the pattern.

In fact, the complete flow-chart for analyzing a symmetry type may be summed up as:

Find the orbifold!

Further Resources

The Symmetries of Things, written with Heidi Burgiel and Chaim Goodman-Strauss, is John H. Conway's account of the Orbifold Theory of Symmetry.

Tessellations: Mathematics, Art, and Recreation, by Robert Fathauer, is a new book suitable for the classroom. Fathauer has many classroom materials available at *mathartfun.com*.

Symmetries of Culture, by Dorothy Washburn and Donald Crowe, contains myriad examples of how symmetry patterns are essential in cultures worldwide.

M.C. Escher: Visions of Symmetry, by Doris Schattscheider, is the definitive mathematical reckoning of this master's beautiful and colorful tessellations.

David Bailey's World of Tessellations *www.tess-elation.co.uk/new-hom* is a comprehensive catalogue of real-world tessellations.

Tilings and Patterns by Branko Grünbaum and G.C. Shephard is a recently reissued classic.

iOrnament (iOs), by Jürgen Richter-Gebert, and *Kaleidopaint* (iOs, Android) by Jeff Weeks are both wonderful pieces of software that build symmetrical intuition, easily, in a deep way.

Jeff Weeks' *Shape of Space* is a fascinating book with many hands-on activities exploring the topology of surfaces and their associated symmetries.

Common Core

Standards for Mathematical Practice

The activity is inquiry based, and the students discuss their reasoning. It is in general alignment with these standards of mathematical practice:

CCSS.MATH.PRACTICE.MP1 Make sense of problems and persevere in solving them.
CCSS.MATH.PRACTICE.MP2 Reason abstractly and quantitatively.

CCSS.MATH.PRACTICE.MP3 Construct viable arguments and critique the reasoning of others.
CCSS.MATH.PRACTICE.MP6 Attend to precision.
CCSS.MATH.PRACTICE.MP7 Look for and make use of structure.

The activities together satisfy:
CCSS.MATH.PRACTICE.MP8 Look for and express regularity in repeated reasoning.

Standards for Mathematical Content

This lesson aligns with the standards for geometry in the fourth and eighth grades, and in high school.

CCSS.MATH.CONTENT.4.G.A.1 Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

CCSS.MATH.CONTENT.4.G.A.2 Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

CCSS.MATH.CONTENT.4.G.A.3 Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

CCSS.MATH.CONTENT.8.G.A.2 Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

CCSS.MATH.CONTENT.HSG.CO.A.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

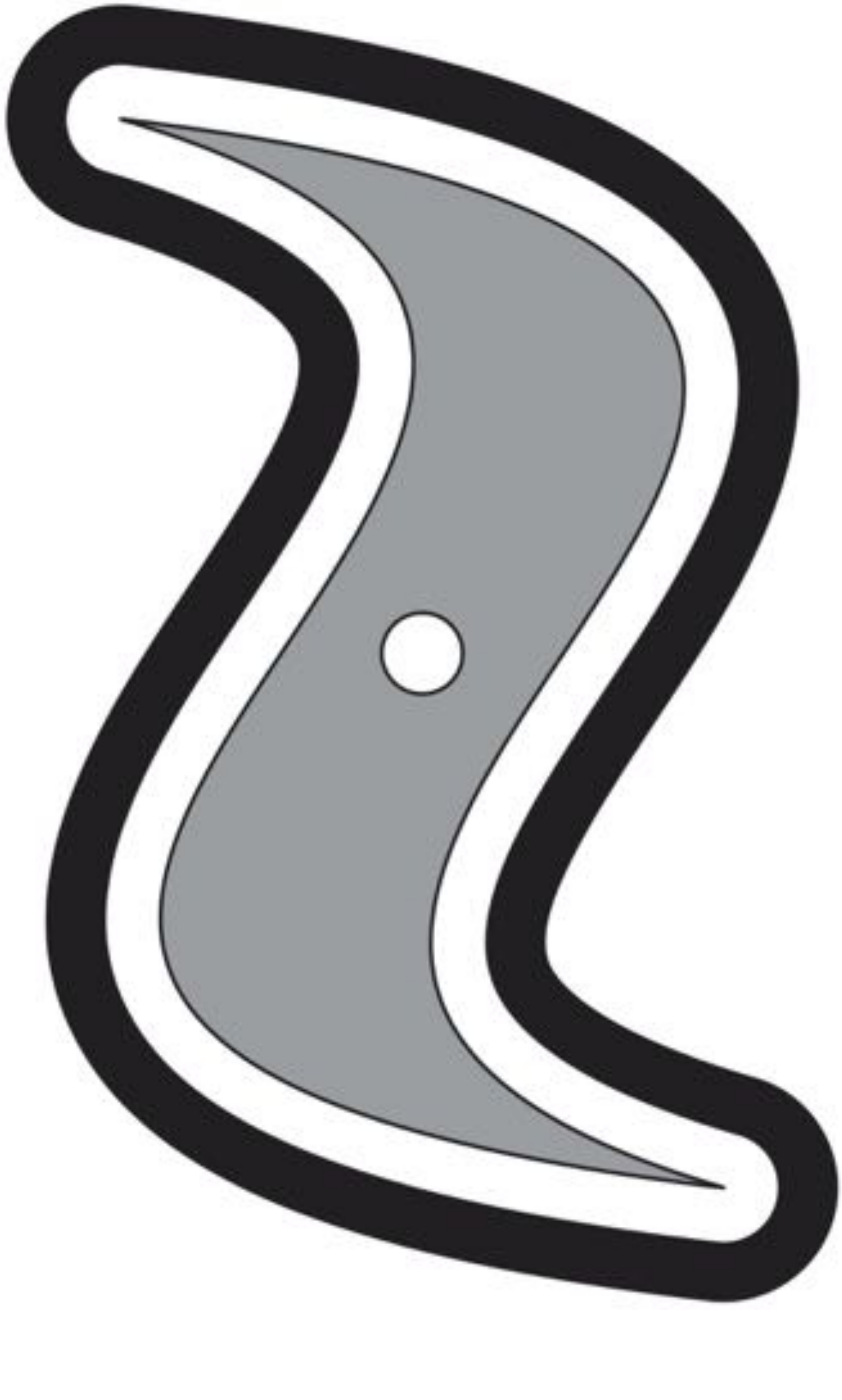
CCSS.MATH.CONTENT.HSG.CO.B.6Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

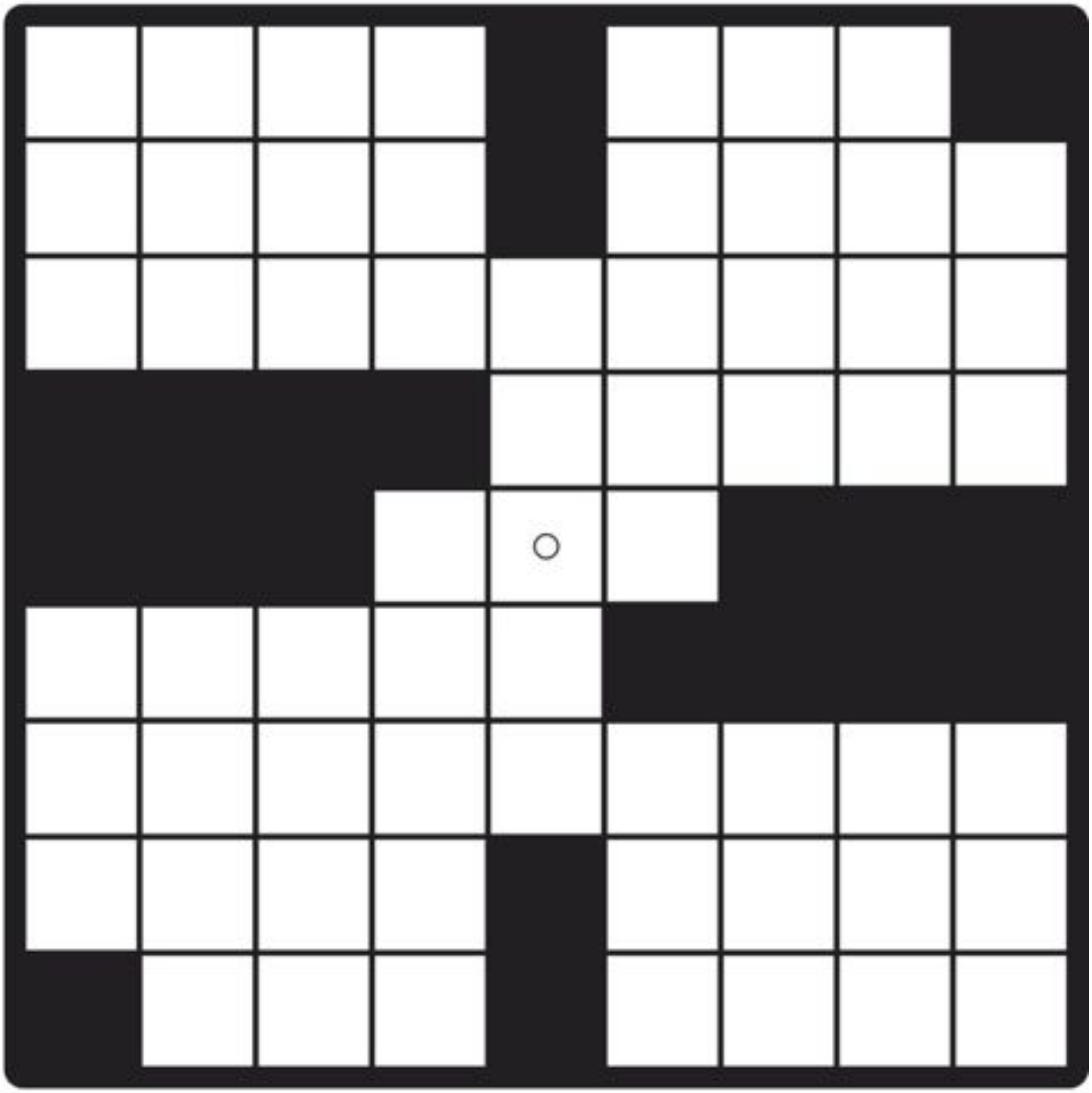
Classroom Materials

Materials to print out for the classroom are on the following pages.


blue
red
red
black
black
Tooti Tooti!


don't cut through the other side!
Tooti Tooti !
don't cut through the other side





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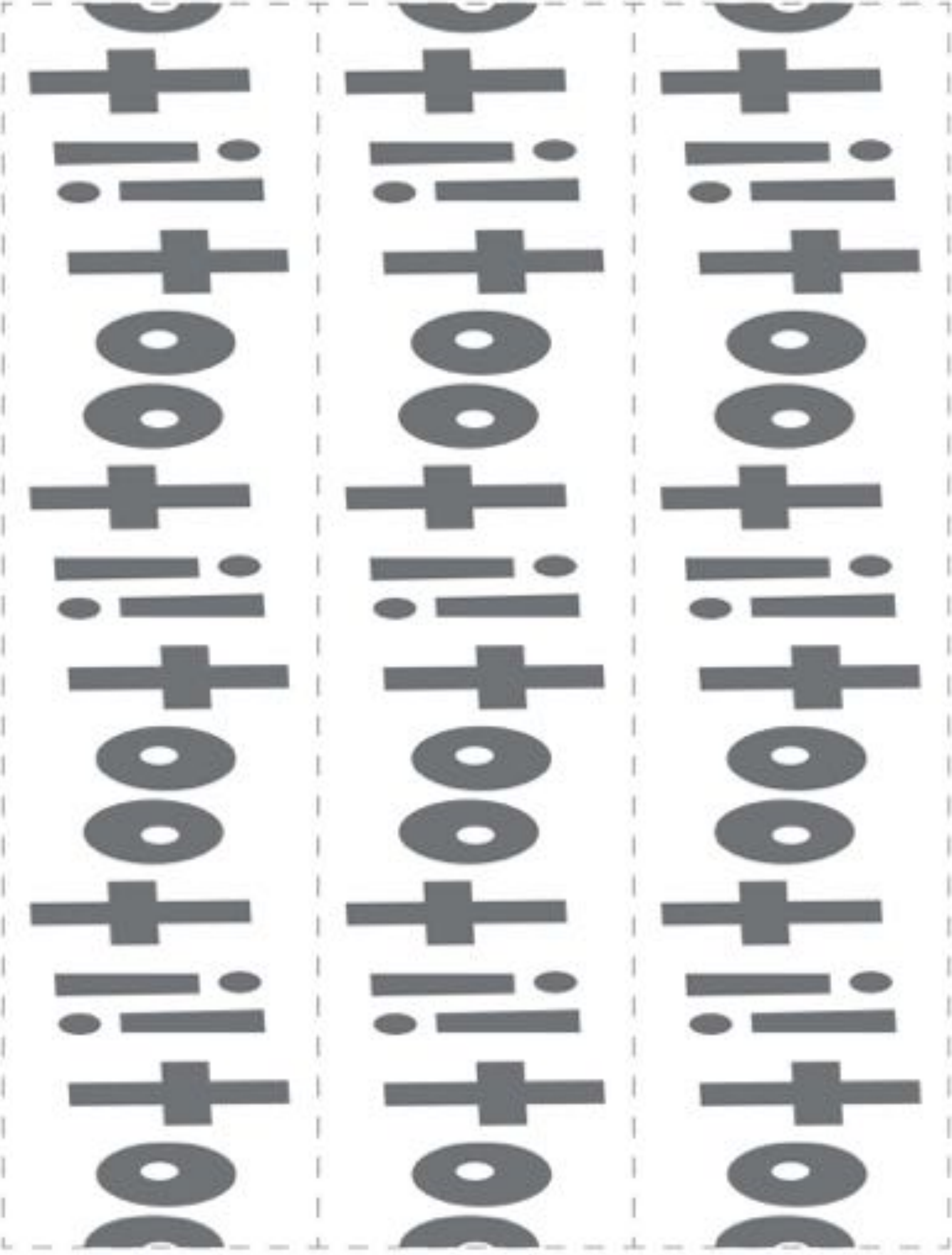

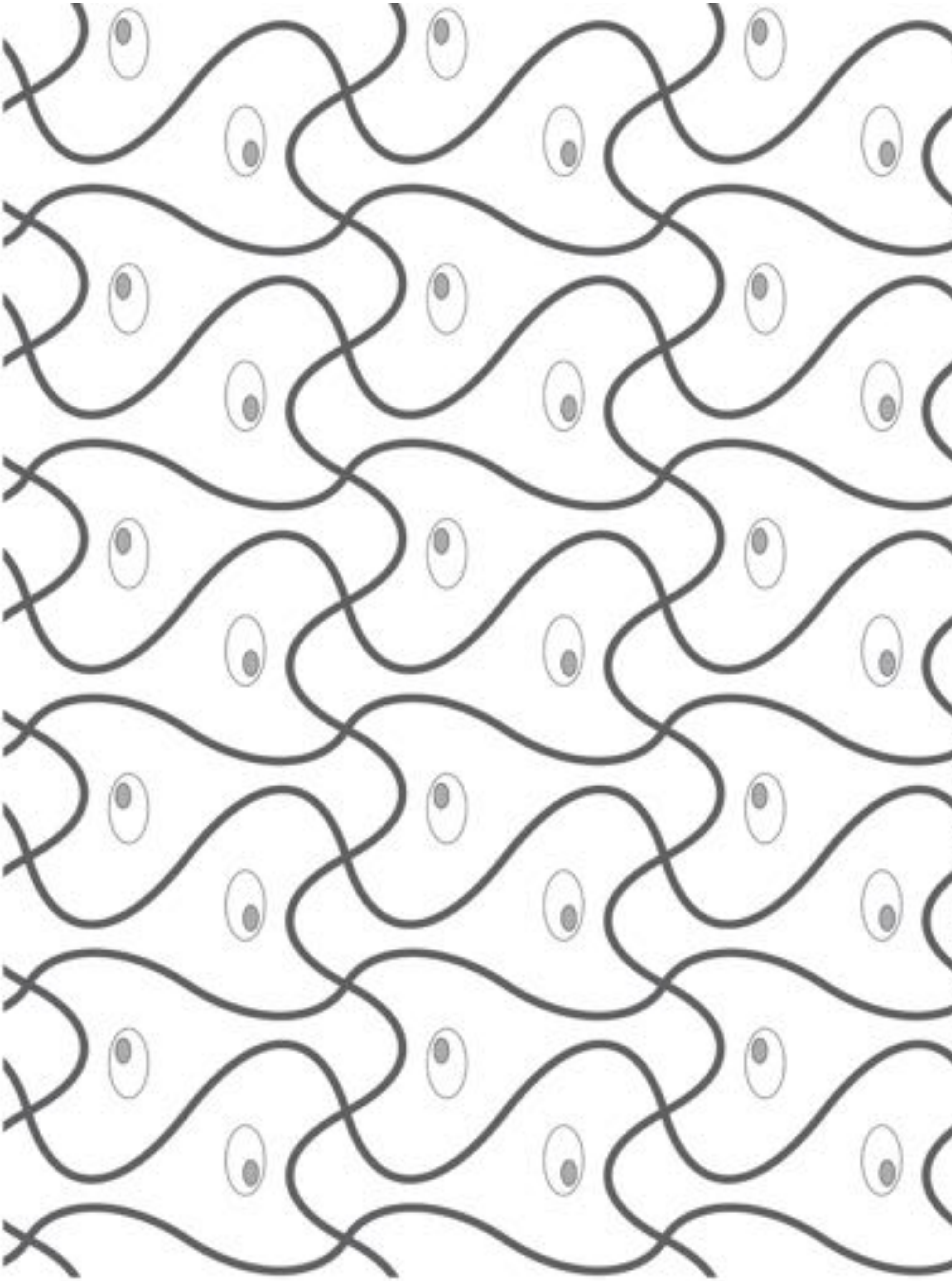

trim and put lots of these together!