TIC-TAC-TOE

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1 Introduction

Tic-tac-toe is a classic game that is commonly played among young children and is known for its simple rules. It is played by two people who take turns marking on X or an O on a 3x3 grid. The first person to mark three of their signs in a horizontal, vertical, or diagonal row is the winner. If both players use an optimal strategy, the game will always end in a draw, and this game would become very boring and repetitive. However, this study will explore two different ways to modify this game. The first way will be to increase the width of the board using an example of a 4x4 board. The second way will be to use a 3D board using the example of a 3x3x3 board. Both cases will determine whether they end in a tie or a win/loss when put under optimal conditions. This study also talks about some of the strategies that the players can use. Furthermore, a formula is derived from the combination of all the example cases discussed in the study and generalizes them into a theorem, which is supported by two different proofs. Finally, a new game is introduced which concludes that the game is actually tic-tac-toe in disguise. Below is a brief history of the origins of tic-tac-toe.

1.1 History

The very first traces of tic-tac-toe go back to Egypt, which has remnants of 3x3 game boards on roofing tiles from 1300 BCE. Other variations included the *terni lapilli* (three pebbles at a time) from the Roman Empire, three men's morris from various parts of Asia, and Picaria from Native Americans. The name "tic-tac-toe" came about when the British name of "noughts (zero) and crosses" was first printed in a scholarly journal in 1858. This was renamed "tic-tac-toe" in the 20th century. Furthermore, this was not only a paper and pencil game, but it was also one of the very first video games, specifically developed by British computer scientist Alexandar S. Douglas in 1952. And in 1975, MIT students developed a perfect tic-tac-toe game using a Tinkertoy computer. This is currently showcased in the Museum of Science in Boston.

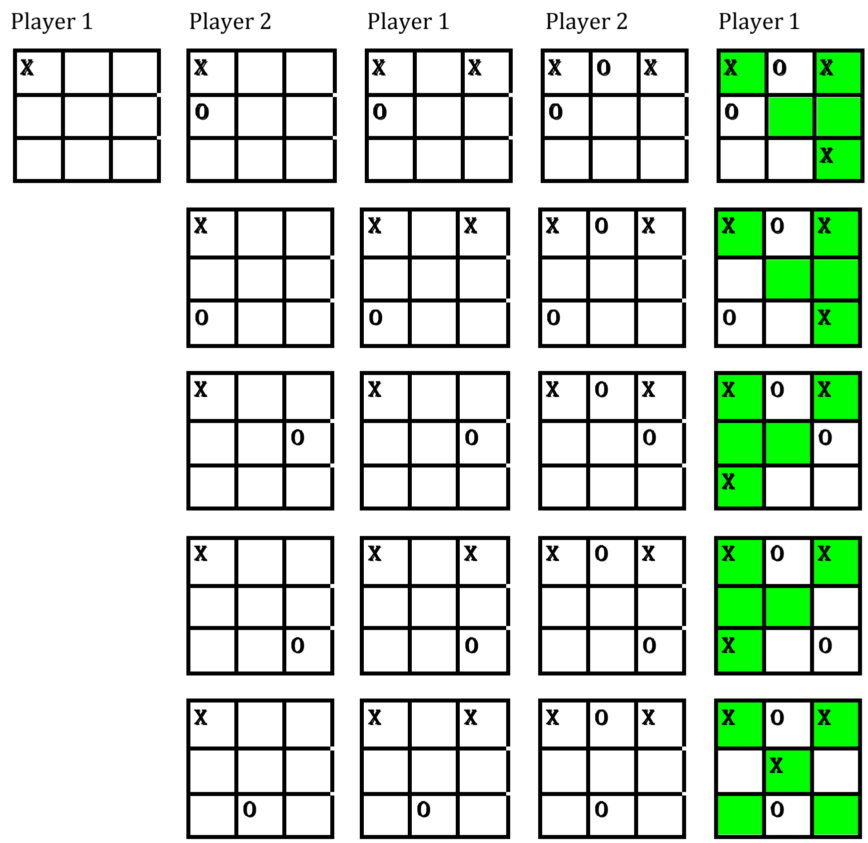
2 Strategy

The basic strategies are as follows:

1. If a player has two in a row, they can mark the third to get three in a row
2. If a player has two in a row, the opponent can mark the third to block the player from getting a three in a row
3. Make an opportunity to form two rows with two marks each that are not blocked - this is known as the fork strategy

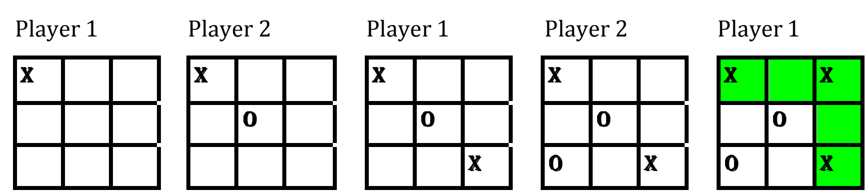
The best chance for a player to win is by using the fork strategy. I will demonstrate the possible fork strategies and the possible blocks that the opponent can use.

If player 2 does not place their first move in the center



Then player 1 will win

If player 2 places their first move in the center:



Then player 1 will win
Or...

X		
	O	
	O	X

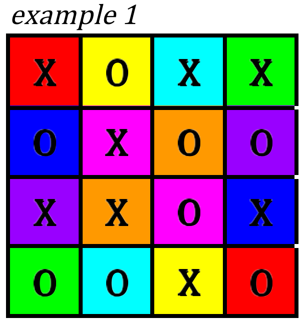
Players 1 and 2 will tie

Even if a player uses the fork strategy, the opponent can block the fork by placing their first move in the center of the board and making the optimal moves thereafter. The fork strategy is only possible if the first player marks their first move in one of the four corners, so if the first player marks their first move anywhere else, the two players will continuously block each other's two in a rows until there is a tie. Therefore, the best play from both players will always lead to a tie.

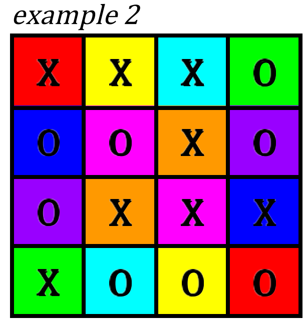
2.1 Increasing the width of the board

We will test out whether increasing the width of the board will allow for a winning strategy. Here is a 4x4 board organized into different colors for each symmetric pair of squares.

example 1



example 2



This illustrates a pairing strategy, where the second player mimics the moves of the first player by placing their moves symmetric to the first player. In the first example, each pair of colors is marked with an X and an O and the game ends in a tie. In the second example, you can see that not every pair of the same-colored boxes are symmetrically paired as shown by the purple and orange boxes. This is an exception to the pairing strategy since the players had to block a three in a row. However, the game still ends in a tie. The reason why this pairing strategy works is that the second player always blocks the potential paths of the first player. This strategy applies to all n^2 boards where n ≥ 4, since the winner of a 2x2 board will always be the first player.

2.2 Using a three-dimensional cube

We will test out a tic-tac-toe game on a 3x3x3 cube. First, we will find out if this game will ever end up in a tie. If this were to be true, then each level of the cube will end up in a draw. Below are the only two configurations of a tied 3x3 tic-tac-toe board discarding symmetries and switching the X's and O's.

X	O	X
O	X	O
O	X	O

X	O	X
O	X	X
O	X	O

We will use the first configuration and use it as the first level of the cube. Below are the three levels of the cube in height order. Let's label each square of the cube based on its level and placement on the board. The squares in Level 1 will be 1a, 1b, 1c, … ,1i (the order going from left→right and top→bottom). The second level will be 2a, 2b, 2c, and so on.

Level 1

X	O	X
O	X	O
O	X	O

Level 2

	O	
X	**X**	**X**

Level 3

X	O	X
X	O	X
O	X	O

The center square of Level 2 could either have an X or an O, but in this case, O is placed. Notice that Level 3 is configured in a certain way that will force a tie. When we start filling in the squares of Level 2, the configurations 1g→2g→3g will be O→X →O, 1g→2h→3i will be O→X→O, and 1i→2i→3i will also be O→X→O in order to force a tie. However, this will form a row of X's meaning that this game cannot end in a tie.

Now, we will use the same original configuration for Level 2. This time, we will place an X instead of an O in the center square of Level 2.

Level 1

X	O	X
O	X	O
O	X	O

Level 2

	X	
	X	
X		X

Level 3

X	O	**X**
	O	
O	X	O

Although this configuration is made to force a tie, a row of X's in 1a→2b→3c will form which once again shows that this game cannot draw.

Now, we will use the second configuration of a tied 3x3 board for the first level and put O in the middle square of the second level.

Level 1

X	O	X
O	X	X
O	X	O

Level 2

O	O	O
	O	

Level 3

X		X
		X
	X	

After making all possible moves to force a tie, we will still end up in a three in a row.

Level 1

X	O	X
O	X	X
O	X	O

Level 2

	X	
X	X	X

Level 3

	O	
O	O	
O		O

Even after putting X in the middle square of the second level, we do not end in a tie. Based on these four cases, it is impossible to end in a tie in a 3x3x3 tic-tac-toe game. If both players play optimally, the first player will always win if they place their first move in the center of the cube.

The winning strategy is as follows:

1. Player 1 places their first move in the center of the cube
2. Player 2 moves anywhere
3. Player 1 moves so that when player 2 blocks his path, player 2 will not be able to form a two in a row
4. Player 2 blocks the two in a row
5. Player 1 moves where it will form two possible three in a rows
6. Player 2 blocks one of the paths
7. Player 1 wins

This is the same fork strategy that is used by two-dimensional boards, except that in 3x3 cubes, this strategy always works.

Likewise, a 4x4x4 cubic game is a first player win. However, the big difference is that the winning strategy of a cubic is much more complicated than the 3^2 board. The first solution was found by Oren Patashnik in 1977 using artificial intelligence. This involved a similar strategy as a 3^2 game where there would be a chain of forced moves until the first player uses the fork strategy when there is an optimal opening. However, there is no elegant solution to this problem, only a brute-force strategy.

2.3 Theorems

1. The total number of winning moves in any n^d boards is:
   $((n + 2)^d - n^d)/2$

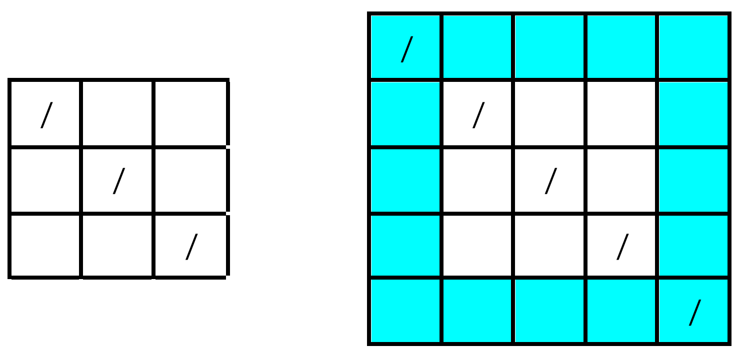
Proof 1:
Suppose that each square of the board is labeled with a set of coordinates. Let's say that $p \in \{1, 2, ..., d\}$ and the set of coordinates are $a_1, a_2, ..., a_d$. In order to form a winning line, $a_p \in \{1, 2, ..., n, increase, decrease\}$. In other words, each of the coordinates either has to be either a constant between 1 and n, go in increasing order, or go in decreasing order, so there are n + 2 possibilities for each coordinate. To find the total possible combinations of winning lines, using this explanation, we will get $(n + 2)^d$ winning lines. However, at least one coordinate has to change, so we have to subtract the total combinations of coordinates that have the same constant, which is n^d. Lastly, since each winning line has two orientations,

the final formula for the total number of winning moves would be $((n + 2)^d - n^d)/2$.

Proof 2:

Imagine an n^d board is surrounded by another layer of squares. That would make the new board including the new layer an $(n + 2)^d$ board. Imagine that the n^d board is cut out so that only the border is left. The total combinations of winning lines of the $(n + 2)^d$ board will mark each square of the border, which has $(n + 2)^d - n^d$ squares. But in order to count each winning line as one, you have to divide that amount by two, where you end up getting the final formula of $((n + 2)^d - n^d)/2$.

To demonstrate this, here is an example of a 3x3 board and a 5x5 board. In this case, n=3 and d=2.



Notice how when I run a line through the 5x5 board, this winning line can be counted as one if I only count the marked squares in the border of the 5x5 board and divide them by two. If I do this for every single winning line, then every square of the border will be marked, and that is how you derive the general formula. In this case, the total number of winning lines will be $((3 + 2)^3 - 3^3)/2=8$.

2.4 Finding Fifteen

Here is another fun game. Let's say there is a list of nine numbers from 1→9. Two players take turns choosing one number from the list without repeating the numbers more than once. The first person to choose three numbers that add up to 15 wins the game. Here are some interesting questions: what makes this game special? What kind of game does this remind you of?

1. Let's list out the possible combinations of three numbers from the list that adds up to 15:
   {1, 5, 9} {1, 6, 8} {2, 4, 9} {2, 5, 8} {2, 6, 7} {3, 4, 8} {3, 5, 7} {4, 5, 6}

   We notice that there are four combinations with 5 in them, so if we were to put these numbers on a 3x3 board, 5 will be placed in the center square. Next, we also notice that 1, 3, 7, and 9 each are in two combinations, so it will go on the edge squares since it can produce two winning lines. Lastly, 2, 4, 6, and 8 will be placed on the corners since each appears in three combinations and the corners have three winning lines.

8	1	6
3	5	7
4	9	2

   Here, we can observe that each row, column, and diagonal of this board adds up to 15. Therefore, the game of finding fifteen is equivalent to tic-tac-toe, meaning that they are isomorphic.

2.5 Conclusion

There are still many unsolved games of tic-tac-toe left in the world. Examples are cases when $n > 4$ and $d > 3$. However, mathematicians came up with interesting theories that brought up general patterns among tic-tac-toe games in more complex and higher dimensions. For instance, Patashnik in 1980 made a conjecture that tic-tac-toe games would end up in a tie if there are more points than winning lines. He also predicted that if the dimension is greater than the length, then the first player will have a definite winning strategy. Although these conjectures are based on a limited number of cases and are merely speculations, these types of ideas continue to enhance the unknowing mysteries of the mathematical world and give insight to a game as simple as tic-tac-toe.

References

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