NATIONAL MUSEUM OF MATHEMATICS

THE 2020 ROSENTHAL PRIZE

for Innovation in Math Teaching

Engagement with Ratio and Proportion:

Building a tool to deepen understanding

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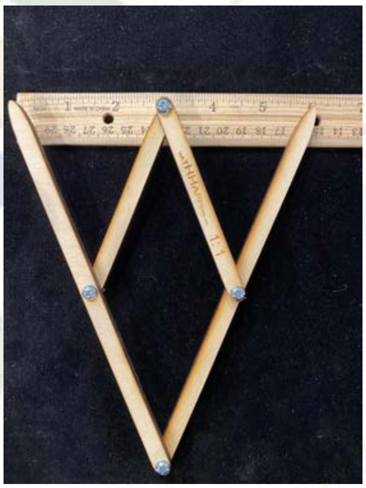
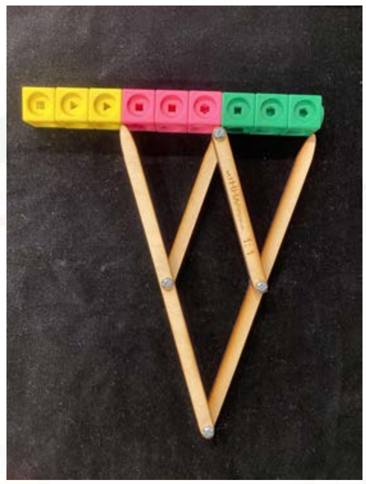
The demonstrations, activities, and discussions in this lesson are intended for use in formal (school or enrichment program) or informal (museum or nature center) settings. The lesson may be useful and effective in any educational framework. The smaller moveable calipers could also be used in a demonstration over Zoom or could be provided in a learning kit. The point of this lesson is to build a tool and then to relate mathematics concepts to concrete objects by using that tool. This lesson does not include TEKS or Common Core or other standardized guidelines.

Introduction

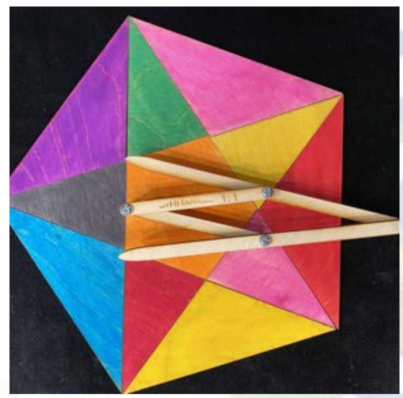
With a teacher set of calipers, either jumbo-sized for a large classroom or the small ones over Zoom, we demonstrate the meaning of 1:1 in a variety of ways: by looking at the device, comparing it to a ruler, and also using the calipers to find midpoints.



3”: 3” or 1:1

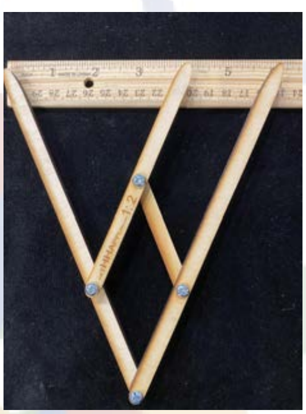


1:1 or ½ and ½

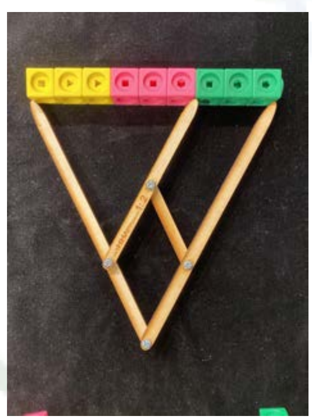


Finding a midpoint

Now we move to 1:2. What does that mean for the points on the ruler? What does it mean if we apply the calipers to an object? The “1” side is ⅓ of the whole, yet the proportion indicates that the other side is twice as large as the left. This is a nice intermediate observation.



4:2 or 2:1

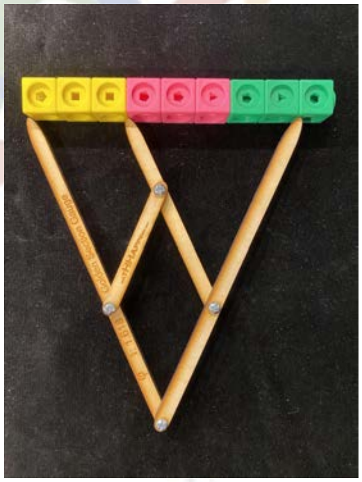


2:1 ratio or ⅔ and ⅓ of a whole

2:1 or 1:2 doesn’t work here

The third set of calipers is made using 1:1.618, to approximate phi, an irrational number and mathematical constant like pi that is structurally integral to the pentagon and five-pointed star and which also can be estimated by taking ratios of successive Fibonacci numbers.

1:1.618 approximately. Note: the calipers are blunt for safety, and there will be errors in the placement.



These calipers are clearly not 1:2, and by counting the blocks, we can see that the proportion is closer to 3:5, which is 1:1.67. This kind of blocks could help decode "mystery calipers."

The 1.618 calipers will correspond well here, and we can construct a proof that the proportion of the triangle side to its base, the pentagon side, is the irrational constant, phi.

With this understanding of the workings of the calipers, and the relationships they describe, we could ask some questions, or elicit questions from the students.

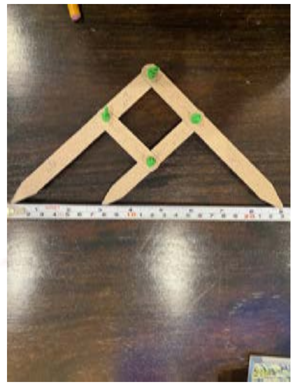
A) How are these calipers similar to one another and does this similarity give us clues to the construction? (Answer: There is a rhombus in the center that keeps two of the arms parallel to each other.)

B) In particular, what makes them work? For high school students we are looking for a proof. For younger students we expect observations about the physical properties. In the 1:1 calipers, for example, the screw and nut on the side are halfway down the arm. They are ⅓ of the way down the arm on the 1:2 and 1/2.618 of the way on the phi set.

C) As an alternative, we can consider doing fewer demonstrations up front and proceeding first to the building.

<u>Activity - Build Your Own 1:2 (or other proportioned) Calipers</u>
Materials:
Empty Cereal Box * Ruler * Scissors * Hole punch * Balloons



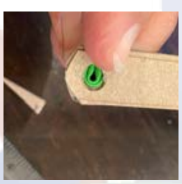
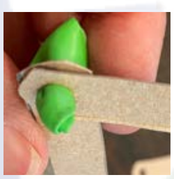


*Cut 4 half-inch strips out of a cereal box.

*Punch a hole in one end, set a second hole 2” from the first (center of hole to center of hole) using a standard hole punch (see photo).

*Set a mark 4” from the second hole for 1:2 (or use some other distance for other proportions) that indicates the location of the point at the end and sketch the end of the arm.
*Cut out the first arm.
*Make a second arm by tracing the first.
*Make a short connector piece with two holes spaced by tracing the original piece.
*Make the center arm the same length as the lower portion of the long arm, but with holes spaced as in the top portion.

Assemble with Balloon Joints
*Using any kind of balloon, new or used and deflated, fold balloon over and pull into holes.

This way of connecting the cardstock keeps the holes aligned with each other and results in a more accurate caliper than brads or other fasteners. The joint moves, but also holds steady.

With calipers built, students can do their own explorations, experiments, and constructions.

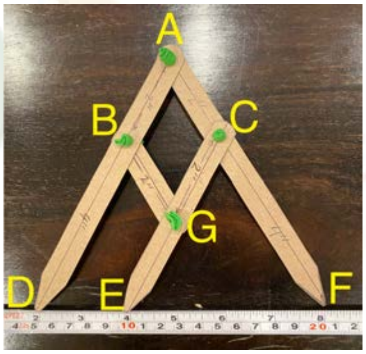
Discussion

We can discuss the geometries we can identify such as parallel lines, equal angles, equal segments, and of course the rhombus in the center.

Proving the function:
The calipers work because they are constructed with a rhombus in the center. This is built into the construction of the calipers when we space the holes, and it guarantees the resulting proportions and holds two of the arms parallel. Lengths CF, CE, and BD are constructed to be equal, and we therefore have similar triangles.

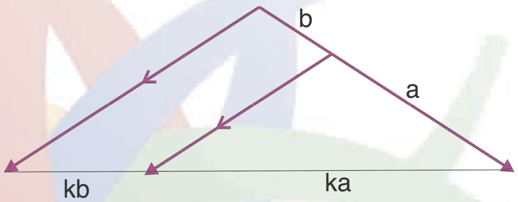
Angle DFA = Angle EFC
Angle DAF = Angle ECF (since ACGB is a rhombus)
Angle ADF must be equal to Angle CEF
Therefore triangle DAF is similar to triangle ECF and CF/AF = EF/DF

That is to say that the proportion of CF to AF is preserved and expressed in the bottom of the caliper EF/DF regardless of the lengths measured by segments EF and DF.



Note that the accuracy of the hand-cut cardstock can be improved by drawing a straight line with a ruler through the two holes so that the line at the bottom indicates the most accurate location on the ruler.

The rhombus we construct as we make the calipers also guarantees that CE will be parallel to AD. So we can also consider and discuss the following: In any triangle, a line that is parallel to one side cuts the other sides of the triangle in the same ratio.



If students select the length of segment *b* to represent "1," then the relative length of segment *a* delivers the *b*:*a* or in particular 1:*a* proportion for the moveable or adjustable side sized by factor *k*, $0<k<2$. Relating triangles EFC and DFA to this construct may make connection to some curriculum results clearer. Finding the range of k might be an interesting investigation for students individually or in a group. Students may also notice that the triangle in the caliper will be equilateral when $k=1$.

*Once the students or activity participants have made one kind of caliper, we can assemble elements of a proof using observations about the design. And we can also ask them how to build a similar device that gives a different proportion, like 1:1, 1:2, or any of their choosing.
*Students could also make a progression like 1:1, 1:2, 2:3, 3:5, 5:8 and see that our proportional calipers made using Fibonacci numbers are approaching phi in the ratio.
*It is also possible to follow the directions without using the ruler to space the holes and set the point: just make one arm of random length and with holes randomly set and then construct the rest of the pieces based on that one randomly set arm. At the end you have a set of calipers set to a proportion — but what is it? Finding that out can be an interesting investigation. Using the line of cubes to detect a likely proportion is one possibility.
*Along the way we can try to use other irrational numbers like the square root of 2 (1.414…), which may or may not be found in structures beyond the diagonal of a square. If we make the calipers we could “look” for that one too. Understanding irrational numbers as numbers existing as finite dimensions, greater than some numbers like 1.4 and smaller than others like 1.5, is another useful outcome for students.

<u>Experiments/Investigations and Discussions</u>

Find examples in objects, geometric constructions, and pictures on-screen, printed, or created. See which proportions work on what structures and which ones don't.

<table>
<tr>
<td></td>
<td></td>
<td>Computer Screen: The cardstock calipers can be safely put on a computer screen, so any variety of images can be explored without printing or other steps.</td>
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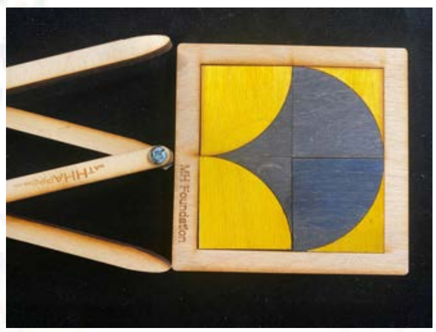
<tr>
<td>
Active Engagement in Teacher Training. Demonstrating a 1:2 proportion using 3 equal tiles</td>
<td>
1:1</td>
<td>Tabletop:
A variety of tabletop explorations can be offered using blocks, printed pictures, everyday objects, or other mathematical puzzles.</td>
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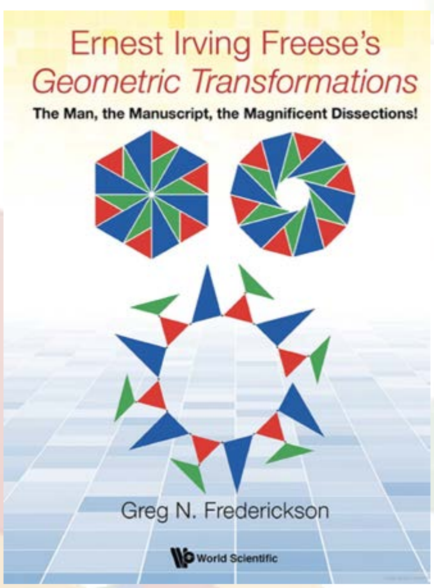
<td>
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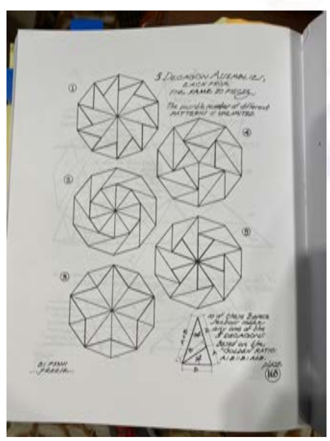
<td></td>
<td>Other options for finding phi:
By making sets of the triangle described in Freese's manuscript, a variety of designs can be constructed that have elements in proportion 1:phi.</td>
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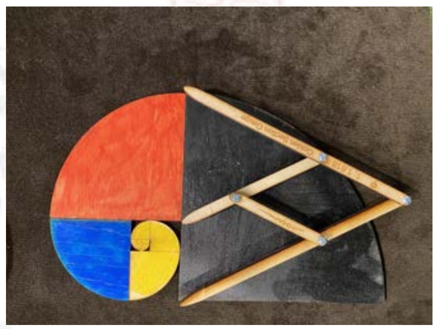
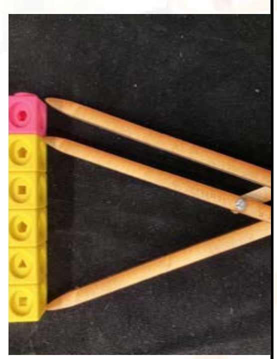
		Some ways to use the calipers on a novel block arrangement, and on one of the ones suggested by Freese.
		These quarter circles are sized to 1.618 and can be assembled into a spiral and other patterns.

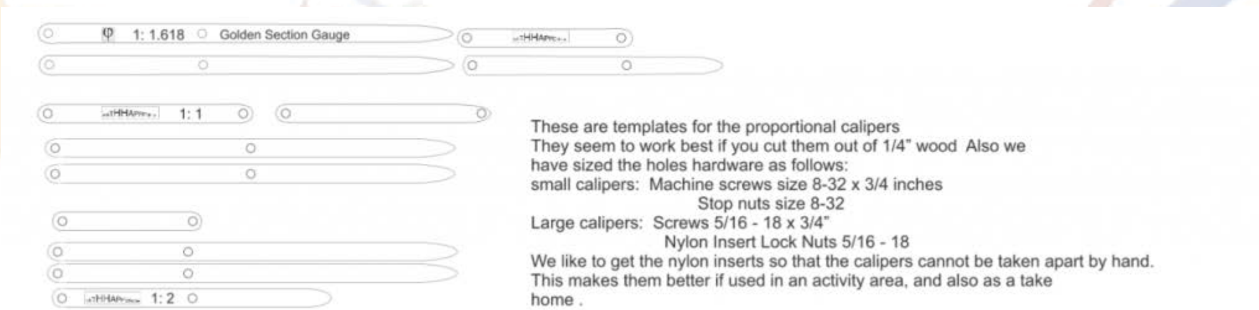
 Close up	 Far Away	 5:1	Exploring “the wild”: Outdoor objects offer opportunities to look for ratios, and extreme versions of the calipers might be imagined and constructed.

Leveraging access to a Makerspace

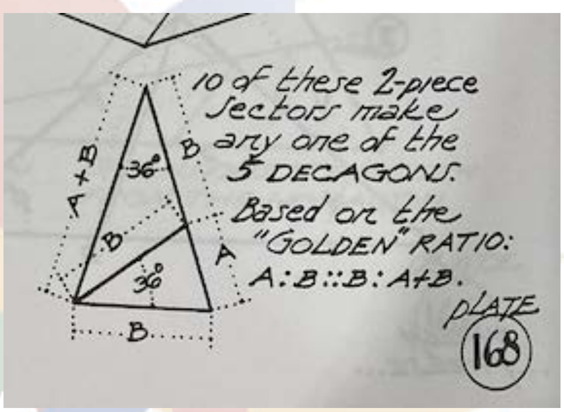
For educators with access to a laser cutter, the following files could be used, or remade in the appropriate software. Hardware can be obtained first and the holes adjusted to work with available screws. Building a durable tool increases engagement.

Design Files to Make Calipers

If students have the opportunity to access a makerspace and use the design software, even laser cutting their own parts, the project aspect can be expanded and general shop skills can be learned and practiced as well. Examining the layout of the holes and the relative geometries of the parts for the three designs provides another way to approach the geometric similarities and differences. [SVG files available here.]



The tiles suggested by Freese can be made on a laser cutter, or cut from cardstock. Creating the design file in graphics or mathematics modeling softwares from Freese's instructions is a great exercise.



From *Ernest Irving Freese's* Geometric Transformations*: The Man, the Manuscript, the Magnificent Dissections!* by Greg Frederickson
World Scientific, 2018