NATIONAL MUSEUM OF MATHEMATICS

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Towers & Dragons

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Grade 8

Task at a Glance

The Tower of Hanoi puzzle and the Dragon Fold have an unexpected connection. The minimum number of moves to solve a puzzle with *n* discs is $2^n - 1$; a strip of paper folded *n* times (which is called a "Dragon Fold") in the same direction will have $2^n - 1$ creases. This connection, however, is even deeper than this correspondence indicates. There is a specific order that the discs in a Tower of Hanoi puzzle must be moved in order to achieve the minimum puzzle solution; this sequence maps directly onto a numbering of the creases in the paper. This connection is mathematically surprising and engaging!

The lesson launches with students solving the Tower of Hanoi puzzle with 2, 3, and 4 discs. They record the sequence of disc moves for each solution and look for patterns. At the end of this stage of the lesson students should be able to see that the puzzle can be solved in $2^n - 1$ steps, that the sequence of moves is palindromic, and that the sequence for $n + 1$ moves can easily be generated from the sequence of *n* moves.

The second half of the lesson introduces students to the Dragon Fold. A strip of paper is folded repeatedly from left to right. After each *n*th fold, they are asked to write the number *n* on every new crease and discover (surprisingly) that this sequence exactly matches the solution sequences for the Tower of Hanoi puzzle! After discussing this correspondence, students are given several other avenues of exploration to complete the investigation.

Lesson Goals

Towers & Dragons is appropriate for an 8th-grade class studying functions, possibly as a launch for such a unit. Students see that a single functional relationship, $f(n) = 2^n - 1$, can model two different situations, where the input *n* can represent the number of discs or the number of folds, and the output gives the number of moves or the number of creases. The lesson also exposes students to a non-linear function in a concrete way.

The lesson also provides multiple opportunities for students to engage in the Standards for Mathematical Practice. Students not only notice patterns but also make arguments as to why these patterns hold. This stems directly from the fact that they are modeling with mathematics the physical activities of solving the Tower of Hanoi puzzle and conducting the Dragon Fold; the physicality of the activities not only gives an opportunity to model, but also cognitively supports students' justifications for the generalizations the students go on to make.

As a unit launch, the lesson is meant to give teachers formative information about their students' background knowledge. Are they flexible in viewing certain parameters as inputs and others as outputs? What vocabulary are they using? How do they conceptualize patterns that they see? How much facility do they have in observing such patterns and writing a corresponding function as an algebraic expression? Without having experienced a formal unit on linear functions, what patterns do they notice when they see tabular data, and how do they think about what 'makes' that data linear or not? Students can start to formulate their own thinking, and later in the unit teachers can promote student agency by building off of the prior knowledge students share here.

Students will:

- Extrapolate data tables to predict values.
- Make conjectures about numeric patterns and justify why those patterns hold in general.
- Create and use mathematical notation to describe physical processes.

Common Core Standards for Mathematical Practice

- *CCSS.MP2 Reason abstractly and quantitatively*: Students will decontextualize results to look for arithmetical patterns and contextualize their results in order to make supporting arguments.
- *CCSS.MP3 Construct viable arguments and critique the reasoning of others*: Students will explain how the patterns they identify in the data associated with the Tower of Hanoi puzzle and the Dragon Fold hold in general.
- *CCSS.MP4 Model with mathematics*: Students will generate data about solving the Tower of Hanoi puzzle and the Dragon Fold, analyze the data, and draw conclusions.
- *CCSS.MP8 Look for and express regularity in repeated reasoning*: Students will recognize how they can apply the same reasoning to explain connections between one iteration of the sequence of moves to solve the Tower of Hanoi and the next or one Dragon Folding crease sequence and the next, and show how this reasoning can be applied inductively.

Common Core Standards for Mathematical Content

- *8.FA.1 Understand that a function is a rule that assigns to each input exactly one output*

Prerequisite Knowledge

Students should be familiar with collecting data in tables and looking for patterns. This activity is meant to be used early in a unit introducing functions. Students may bring a wide range of prior knowledge and experience to such a unit, and the tasks here will allow teachers to assess that knowledge formatively for use throughout the unit.

Time Required

Preparation Time: 10-20 minutes, depending on how much paper cutting is done by the teacher in advance, versus allowing students to set up the activity by doing their own paper cutting
Class time: 90 minutes

Setup

Materials

- 1 pair of scissors per student
- 5 strips of paper about one inch wide, cut lengthwise from a standard sheet of blank paper: These can easily be cut out by students during the lesson or as class starts. Alternatively, adding machine paper can work well—each student can be given a five-foot strip, which they can then cut into shorter strips for the investigation.
- Discs for the $n = 2$, $n = 3$, and $n = 4$ Tower of Hanoi puzzle: See "Tower of Hanoi Circles" at the end of this document. Cut out the three sets of circles for each student, paper clip them, and put them in plastic bags to be handed out in class, or have the students do the cutting themselves during class.
- 4 Markers or crayons of different colors for each student (optional): These can be used in Part 2b when assigning numbers to the creasing pattern.
- A physical model of the Tower of Hanoi to model the rules of the puzzle at the launch of Part 1 of the lesson (helpful but not required): You can also find many Tower of Hanoi applets

online to project on a screen—for example, https://www.mathplayground.com/logic_tower_of_hanoi.html

Task Directions

Part 1: The Tower of Hanoi [35 minutes]

- Have students cut out the discs (see cutouts at the end of this document). On a blank sheet of paper, have them draw circles A, B, and C, each about the size of the cutout of the largest disc (disc 1 of the $n = 4$ game discs). The A, B, and C circles are the 'Towers' upon which the numbered discs are placed.
- Explain the rules of the Tower of Hanoi:
    - In the initial state, discs are placed in a stack on circle (Tower) A ordered by size. The largest disc (disc 1) will be on the bottom and the smallest will be on top. *Emphasize that for any number of discs, in this lesson, the largest disc is always labeled disc 1*. (Aside: This numbering system will pay off later in the lesson when students number the creases as part of making the Dragon Fold; obviously don't mention this to your students here!)
    - Only one disc may be moved at a time.
    - Smaller discs can be placed on top of larger discs; larger discs cannot be placed on smaller discs.
    - The goal is to move all of the discs from Tower A to Tower B or Tower C.
    - As a class, complete the puzzle with 2 discs. Have the class confirm that the puzzle here can be solved in no fewer than three moves. Also explain that the sequence of moves can be written '212,' with the smallest disc labeled 2 and the largest disc labeled 1.
        - One issue that might get raised is whether the notation for indicating the disc movements needs to not only include the number of the disc being moved, but also the name of the tower the disc is being moved to. Recording this additional information is not necessary because, after the first move, every move is either forced or obvious, but that might take some discussion to sink in. For example, you might let students create their own notation first and then ask, "Do we really need to say *where* disc 1 is going when we write out this sequence?"
    - Have the students individually attempt the Tower of Hanoi puzzle with three discs for 2 or 3 minutes. Again, emphasize that the largest disc is labeled 1. After you have assessed that everyone understands how to move the discs, put students in pairs and give them the *Tower of Hanoi Puzzle Handout*.
- After students complete the handout, discuss:
    - What did you notice about the numbers in the table?
    - Is there consensus about the fewest number of moves for 3 discs? 4 discs? Can they generalize and predict the fewest moves for a tower of any size?
    - What about the sequence of moves for a particular number of discs? Do you see a pattern in the sequence of moves? How can you use one set of moves to get the next set?
    - Try to put into words an argument that the $n = 2$ puzzle cannot be solved in fewer than 3 moves.
    - Imagine trying to solve the $n = 3$ puzzle in 6 moves. Can you explain why this is impossible? Consider the sequence of moves in your argument.

Teacher Notes on Part 1

- Students are usually adept at recognizing that it takes 7 steps to move 3 discs in the Tower of Hanoi puzzle. As you circulate amongst pairs, push students to think about whether they can do better. If not, why not? These conversations will help set up the full group discussion.
- Pairs are likely to finish at different times. If students correctly solve the $n = 4$ puzzle you can ask them:
  - What would be the sequence of moves for $n = 5$?
  - Predict the number of moves it would take to move 10 discs. Be prepared to share your thinking!
  - How could you change the Tower of Hanoi puzzle's rules to create a new puzzle?
- The pattern in the data for fewest moves can be recognized in multiple ways:
  - **The finite differences in the outputs are 2, 4, and 8; these differences double.** Using finite differences is a great way to tell whether a function is linear or not, something that you might or might not point out to students depending on what the group has studied previously.
  - **To get the next output, double the previous output and add 1.** 1 → 1x2+1 = 3; 3x2+1 →7; 7x2+1→15. This observation connects perfectly to the patterns of moves in the table. For example, the pattern 212 (for $n = 2$) is of length 3. This can be used to generate the pattern of moves for $n = 7$ as follows: First, relabel the discs, so 212 becomes 323. Then write 323 twice with a 1 in the middle, or, 3231323. So, to move three discs, we need to move the 2 smaller discs in 3 moves, move the big disc, then the 2 smaller discs again: 2x3+1.
  - **Each output is one less a power of 2.** This observation can lead students to recognize that the formula is $f(n) = 2^n - 1$. This formula is consistent with the 'doubling and adding 1' observation, a connection that can be made or skipped based on student background, the teacher's judgment, and time constraints! Suppose we take a number of the form $2^n - 1$, double it, and add 1: $2(2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1$.
- The argument that the minimum number of moves to solve the Tower of Hanoi puzzle is $2^n - 1$ is inductive. To make this kind of inductive argument accessible for middle schoolers, focus on arguments with specific numbers of discs. For example, it's evident that we need one move to move a single disc and 3 moves to move 2 discs. To move 3 discs, the first 2 discs must first be moved (3 moves) — note here that to move the largest disc, all other discs must be on a single 'Tower' so all 3 moves really are required. Now move the large disc (1 move); then the other 2 discs again (3 moves). So we see that it takes at least 7 moves to move three discs. And so forth.

Part 2a: Folding the Dragon [20 minutes]

- Have students take a blank piece of paper and cut it into 5 long strips, each about 1.5 inches wide. Model the fold that produces a dragon curve:
  - Fold the strip so the left edge maps on to the right edge of the strip
  - Without unfolding, repeat — fold the (new) left edge to the right edge of the half strip.
  - Fold in this way a total of four times, always left to right.
  - Unfold the strip so students can see the resulting 'curve' that's produced; see Figure 1 at right.
  - What do they notice and wonder? What questions can they pose?

Figure 1

- After recording students' noticings, wonderings, and questions, guide them towards the question: How many creases will a strip have when folded this way *n* times? Have students pair with a partner; encourage them to each make the folds and then compare answers. Some will come to the conclusion quickly; push them to explain why the pattern holds in general.

Folds	1	2	3	4	5
# of Creases	1				

- Discuss as a class — how do we know that after *n* folds we will have $2^n - 1$ creases?

Teacher Notes on Part 2a

- Students are likely to quickly recognize that the $2^n - 1$pattern holds for creases as a function of folds. Push for students to explain their thinking about the pattern in terms of the physical motions of the paper (and then have these students share their thinking in the full group). For example:
  - Each time the paper is folded, the number of layers of paper is doubled. The second fold creases two layers, so the table entry for creases increases by 2; the third fold creases 4 layers, so the table entry for creases increases by 4. Hence the finite differences in the table are 1, 2, 4, 8, etc.
  - The initial strip can be thought of as a single rectangle with no creases. Each fold doubles the number of rectangles on the strip. There is always one less crease than rectangle, so the number of creases in the table is 1-1, 2-1, 4-1, 8-1, 16-1, etc.

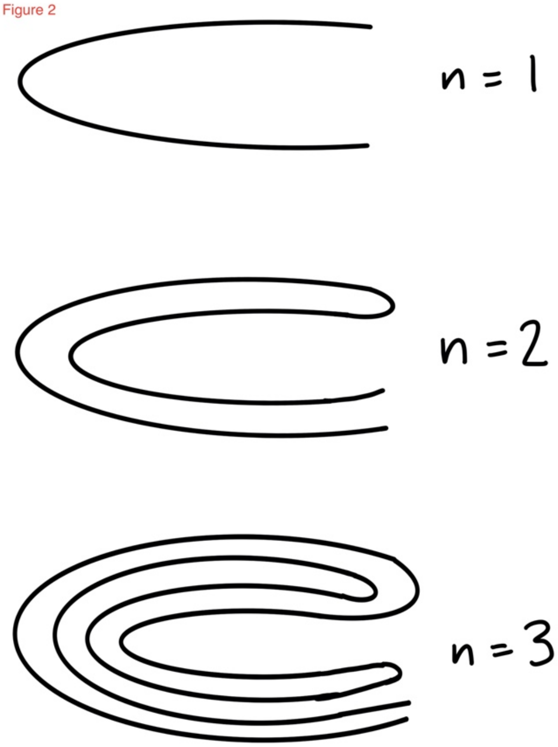
Part 2b: Labeling the Creases [35 minutes]

- Have students fold the strip in the same manner as before, but with the following additional instructions. (Also, tell students they must not speak again until you say so; you want to give all the students the opportunity to experience the big 'reveal' of the lesson for themselves!)
  - Fold the strip once. Unfold and label the crease 1 (if you have crayons or markers, students might also color this crease with 'color 1').
  - Fold the strip twice this time. Unfold and label the two new creases 2 (and highlight them with color 2; etc. in subsequent steps).
  - Fold the strip three times; unfold, new creases labeled 3.
  - Fold four times, unfold, label the remaining creases 4.
- What do you notice?(!) What is it that we need to prove? (The hope here is that students will anticipate some of the questions that follow.)
- Ask students to have a paired discussion, considering the following questions (to be written on the board, or handed out; also see 'Dragon Fold Handout' below; this handout includes a table where students write down the crease sequence for $n = 3$ and $n = 5$ before these questions are posed):
  - Why does it make sense that the highest crease number that appears interleaves the smaller crease numbers?
  - What is the interleaving of the highest number in the sequence analogous to in the solution sequences of the Tower of Hanoi puzzle? Why does that interleaving make sense?
  - The sequence of creases for $n = 3$ is 323 1 323. The sequence of creases for $n = 2$ is 212. So we can get the $n = 2$ sequence from $n = 3$ by taking the right half of the $n = 3$ sequence (323) and subtracting 1 from each number in that left half. Why does that make sense?
- Wrap up the session by having students share their thinking on these questions; see extensions below if time permits!

Teacher Notes on Part 2b

- Consider drawing Figure 2 on the board.

Figure 2

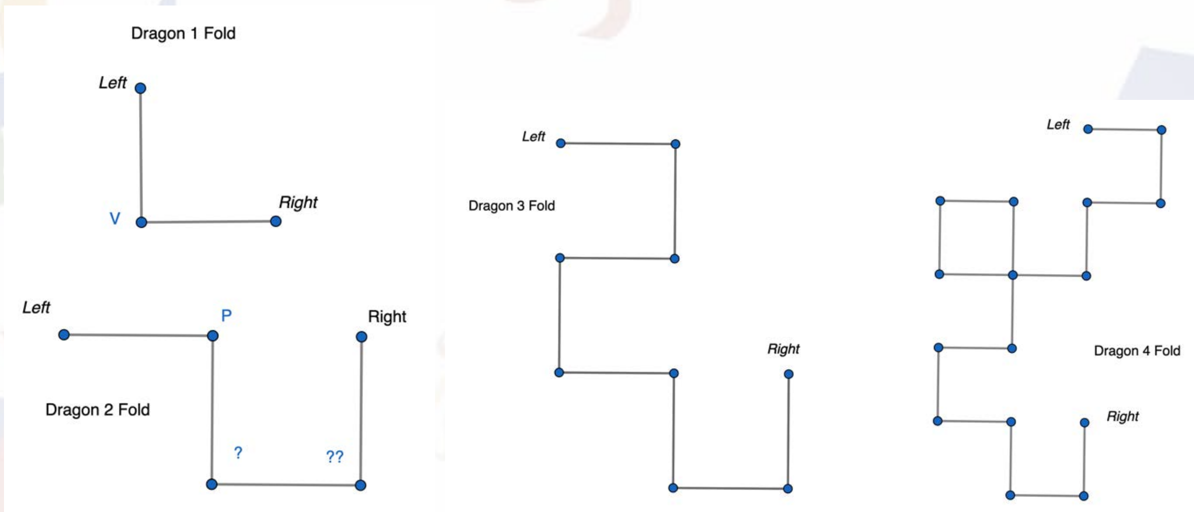


    - As students articulate their answers to the question, you might ask them to come to the board and use these drawings to support their explanations, labeling creases on the figures as they speak.
    - At the same time the students listening to these arguments might 'fold along' to 'act out' the representations on the board.
- Why does it make sense that the highest crease number that appears interleaves the smaller crease numbers?
    - Consider the $n = 2$ case in Figure 2. At this point crease 1 is on the right and the two crease 2's have just been formed on the left. When folding again, we can see that the four crease 3's will be formed at the midpoint of each 'layer,' such that these will alternate with the places for crease 1 and the crease 2's.
- What is the interleaving of the highest number in the sequence analogous to in the solution sequences of the Tower of Hanoi puzzle? Why does that interleaving make sense?
    - This interleaving indicates that every other move in solving the Tower of Hanoi puzzle is made with the smallest disc.
    - The smallest disc (S) clearly must be moved first, and there would be no reason to move the S disc two turns in a row, so we always move another disc directly after moving the S disc.
    - When a disc that is not S (call it X) is moved, it must be moved onto the tower that S is NOT on (by definition of S, X is larger than S). But now one tower has as its highest disc S, another X, and, if the third tower is not empty, a disc larger than S and X. So the next turn must be moving S! And so forth.
- The sequence of creases for $n = 3$ is 323 1 323. The sequence of creases for $n = 2$ is 212. So we can get the $n = 2$ sequence from $n = 3$ by taking the right half of the $n = 3$ sequence (323) and subtracting 1 from each number in that left half. Why does that make sense?
    - On the $n = 3$ drawing in Figure 2, label the points corresponding to the 3, 2, and 3 creases on the right side of the strip; label the points on the $n = 2$ case that correspond to the 2, 1, and 2 creases. We can see that when we remove the 'left' part of the $n = 3$ drawing, it becomes the $n = 2$ drawing once we subtract one from the 3, 2, and 3 labels.
    - Of course, another way to derive the $(n - 1)$th sequence from the nth sequence is to simply strip (pun intended) all of the n's from the sequence: 4342434 1 4342434 → 323 1 323 (Moving in the other direction, of course, involves alternately inserting n's, which we know makes sense from the argument above.)
- Students might notice that the creases due to the nth fold are always on the left side of the drawings in Figure 2, and there are 2^{n-1} such creases; there are $2^{n-1} - 1$ creases on the right; $2^{n-1} + (2^{n-1} - 1) = (2^{n-1} + 2^{n-1}) - 1 = 2^n - 1$.

- Another puzzle for students to consider — could you draw the n = 4 case in Figure 2? Can you describe an algorithm for doing so (beyond folding a strip and looking at it carefully)?

Extension Section

When students first observe the Dragon Fold, they are likely to make many observations that are not explored in the main lesson. These suggestions are great opportunities to promote student agency and go deeper. One particularly interesting question to explore is the pattern of creases that are valley folds versus peak folds.



Here are the sequences of creases labeled V for Valley and P for Peak:

Folds	1	2	3	4
PV Sequence	V	PVV	PPVVPVV	PPVPPVVVPPVVPVV

- What patterns do you notice in the table?
- Given the *n*th sequence, can you give a rule for generating the next sequence?
- What do you notice about the middle crease in all sequences? Why is this the case?
- Unlike the previous sequence we analyzed, these sequences are not palindromic. What is the relationship between the left and right sides of each sequence? What explains this phenomenon?

As you explore peaks and valleys, it's natural to share an online video like this one (https://www.youtube.com/watch?v=UBuPWdSbyf8) that demonstrates why this called the Dragon Fold!

Teacher's note: A possible explanation of the Peak/Valley sequence pattern — Consider the transition from the 3rd to the 4th iteration (see the table below):

- After 3 iterations the sequence of creases is PPVVPVV;
- In the 4th iteration, two imprints of the 3rd iteration are left on the strip;
- On the right side of the strip the third iteration sequence is imprinted (italicized below);
- On the left side, this sequence is reflected (the order of the P's and V's is reversed) and inverted (now Peaks and Valleys interchange), so PPVVPVV reflects to VVPVVPP, and then inverts to **PPVPPVV**. So after four iterations we have PPVPPVV V PPVVPV.

V	PVV	*PPVVPVV*	**PPVPPVV** V *PPVVPVV*

Other possible extensions students may raise from first observing the Dragon Fold:

- What if we fold from the right rather than the left? Or if we alternate directions? Or fold randomly from the right or left?

Tower of Hanoi Extensions

- What if there were four towers? *N* towers?
- What if there were two discs of each size, and discs of equal size could be placed on top of each other, but otherwise the old rules apply?
- Is there a sequence of paper folding creases on a strip that will produce the same sequence for one of these Tower of Hanoi extension puzzles?

Acknowledgements
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Materials for Copying:
The Tower of Hanoi Puzzle Handout, Dragon Fold Handout, Tower of Hanoi Circles

The Tower of Hanoi Puzzle Handout

(a) Place the discs 1, 2, and 3 on circle A of your Tower of Hanoi puzzle. Try to move the three discs to either circle B or C in as few moves as possible. Remember:

- You can only move one disc at a time.
- You can never place a disc on top of a smaller disc (for example, you can place disc 3 on disc 1, but you cannot place disc 1 on disc 3).

Each partner should try the puzzle. When you get a number of turns, share it with your partner — keep trying to improve so that you can complete the puzzle in as few moves as possible.

(b) Once you believe you have moved the three discs in the fewest moves possible, record that number in the table. Also record the sequence of moves in the table.

# of discs	1	2	3	4
Fewest # of moves	1	3		
Sequence of moves	1	212		

(c) Do you see any patterns in the table? What do you predict is the fewest number of turns it will take to move four discs? What would you predict for a sequence of moves? After you make a prediction, try it out to see if it works — if it does, record your answers in the table.

Record your thinking here!

Dragon Fold Handout

1. Complete the table

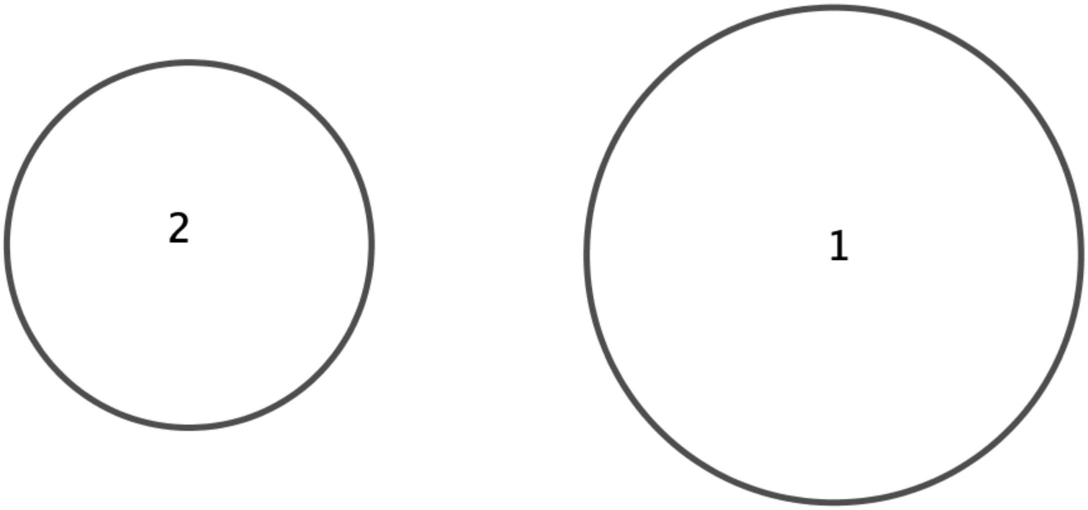
Number of folds	Crease Sequence
1	1
2	2 1 2
3	
4	4342434 1 4342434
5	

2. Why does it make sense that the highest crease number that appears interleaves the smaller crease numbers?

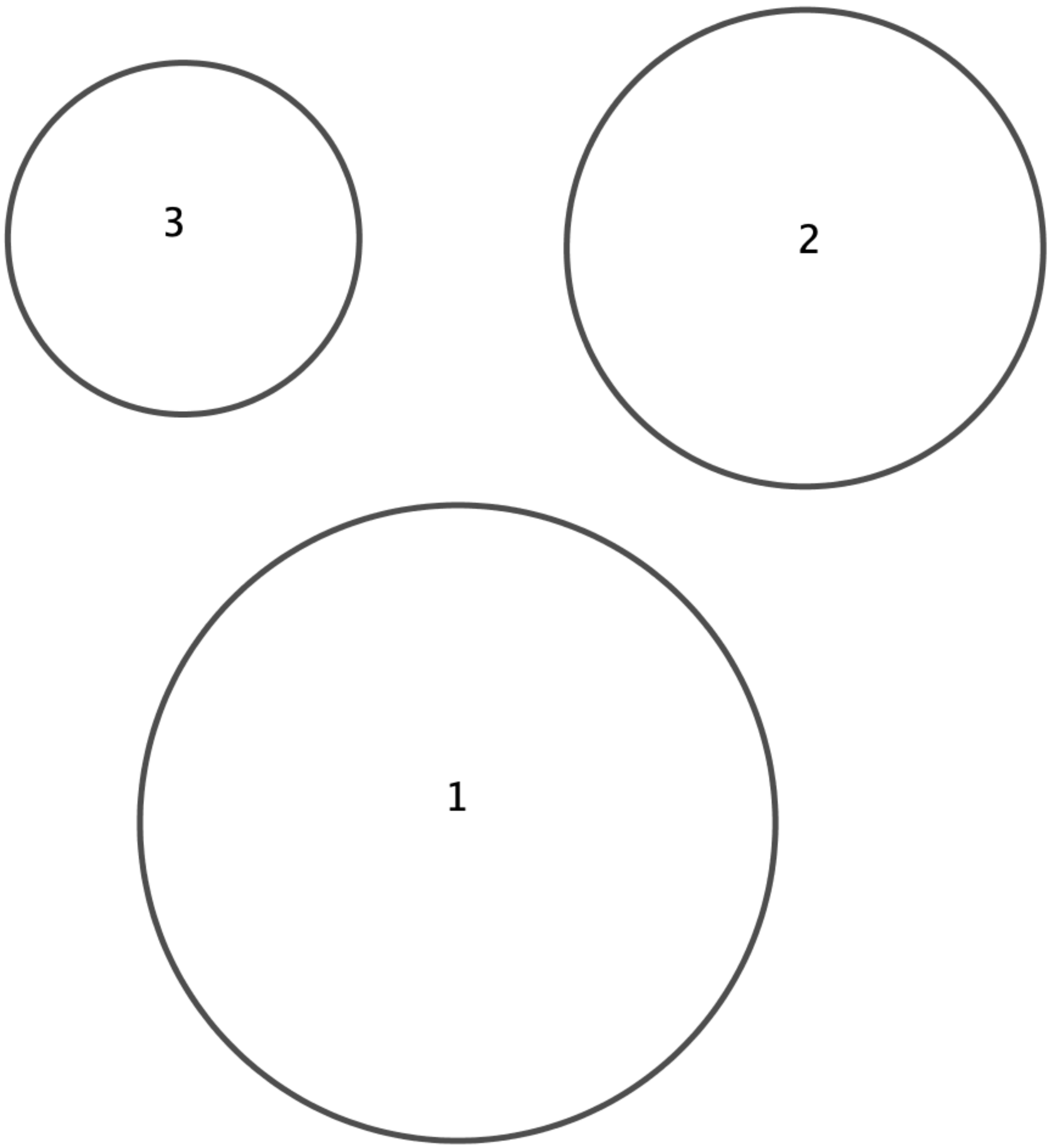
3. What is the interleaving of the highest number in the sequence analogous to in the solution sequences of the Tower of Hanoi puzzle? Why does that interleaving make sense?

4. The sequence of creases for n = 3 is 323 1 323. The sequence of creases for n = 2 is 212. So we can get the n = 2 sequence from n = 3 by taking the right half of the n = 3 sequence (323) and subtracting 1 from each number in that left half. Why does that make sense?

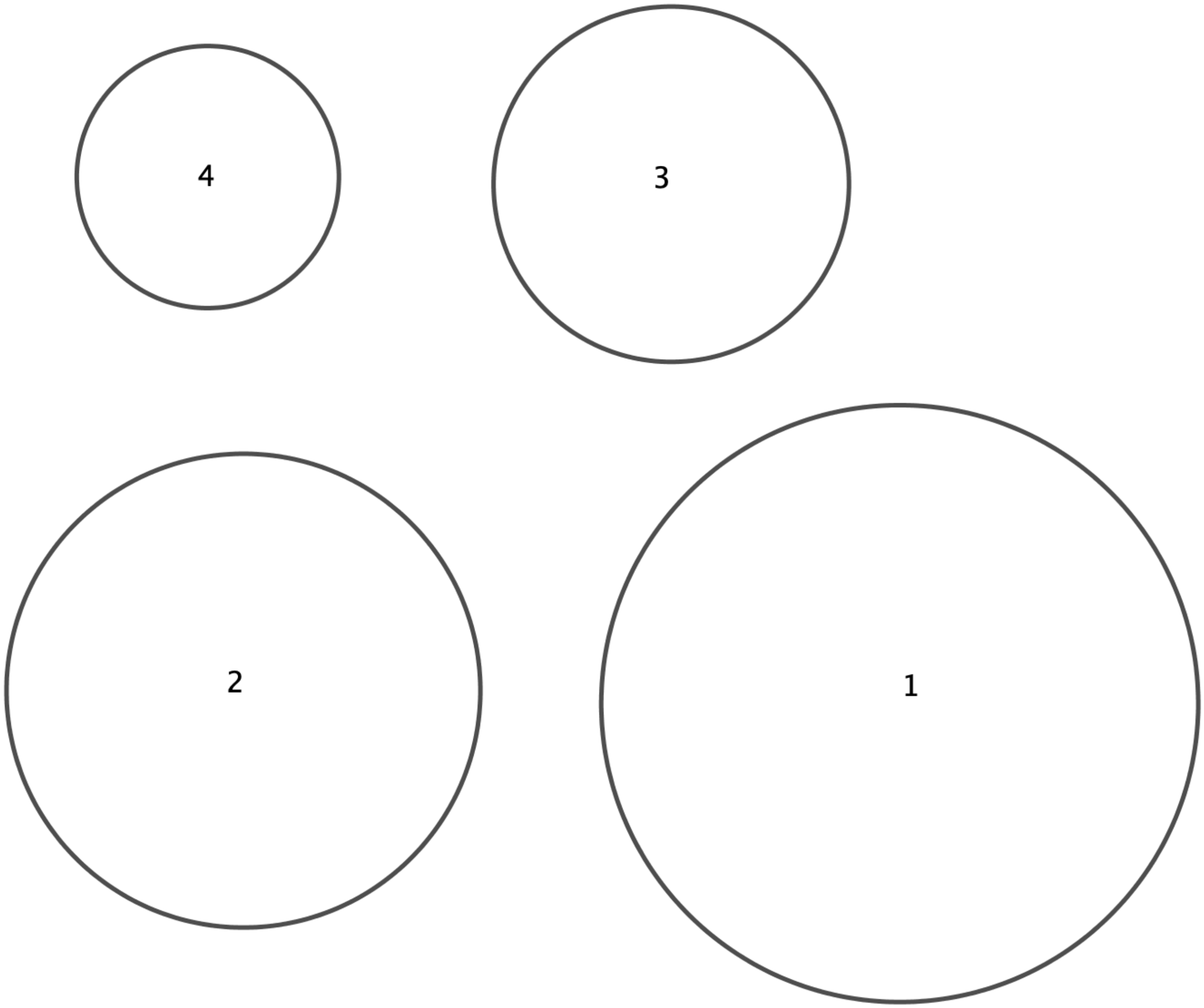
Tower of Hanoi Circles
For the $n = 2$ game



For the $n = 3$ game



For the $n = 4$ game


4
3
2
1

Tower of Hanoi Puzzle Answers

# of discs	1	2	3	4
Fewest # of moves	1	3	**7**	**15**
Sequence of moves	1	212	**323 1 323**	**4342434 1 4342434**

Dragon Fold Handout Answers

Number of folds	Crease Sequence
1	1
2	2 1 2
3	**323 1 323**
4	4342434 1 4342434
5	**545354525453545 1 545354525453545**