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Building the City of Numbers An Exploration of Unique Prime Factorization

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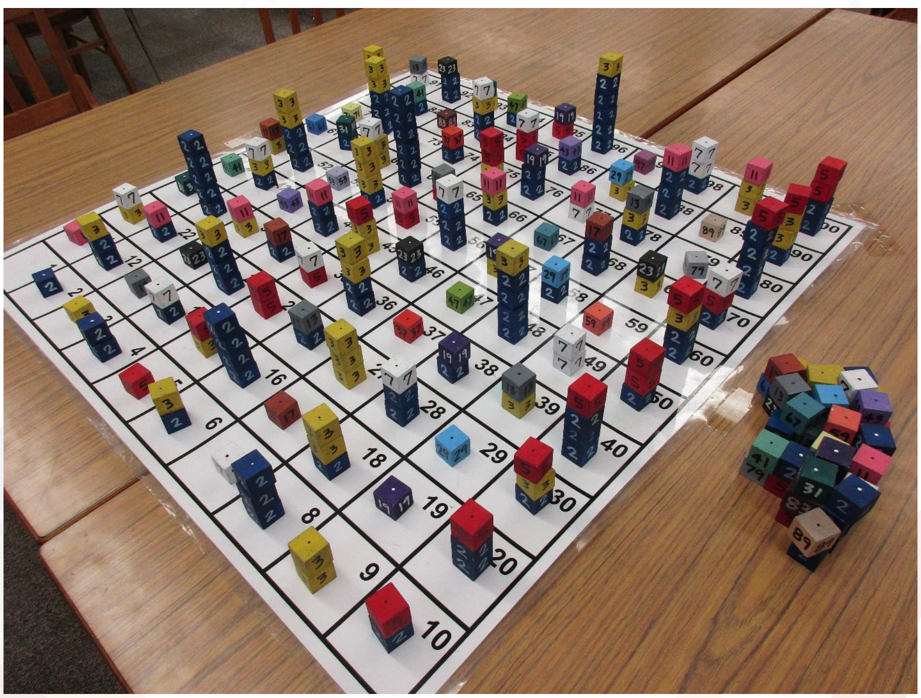


Lesson Plan

Grades 4-8

Contents

Lesson Goals, Practices and Content

Lesson Goals

- Create prime factorizations of the counting numbers 2-100
- Compare prime factorizations of the counting numbers 2-100
- Understand that a number's prime factorization is unique
- Understand that a number's unique prime factorization predicts that number's factors
- Discover patterns among the prime factorizations of the counting numbers 2-100

Standards for Mathematical Practice

- Make sense of problems and persevere in solving them
- Reason abstractly and quantitatively
- Construct viable arguments and critique the reasoning of others
- Look for and make use of structure

Standards for Mathematical Content

- CCSS 4.OA.4: Find all factor pairs for a whole number in the range of 1-100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range of 1-100 is a multiple of a given one-digit number. Determine whether a given whole number in at the range 1-100 is prime or composite.
- Interestingly the Common Core State Standards includes the ability to distinguish between prime and composite numbers in the same standard as finding factor pairs, but no attempt is made to connect these two, seemingly, isolated instructional strands. No other mention in the common core is made of primes and composite numbers. This lesson attempts to bridge this gap by aiming instruction at the goal: Use a number's prime factorization to understand and analyze a number's factors.
- Understand prime and composite numbers according to an atom/molecule metaphor where primes are thought of as the most elemental of numbers (atoms) that are used to construct, through multiplication, more highly divisive numbers (molecules).

Materials

Classroom-Ready Poster

- You will need one 36” by 36” poster with the numbers 1-100 written in a 10 by 10 array starting with one in the lower left and proceeding to 100 in the upper right. It is helpful if the poster is laminated for durability as the poster is to be taped down. This poster is one document found at this link.

Prime Number Blocks

- You will need 279 1-inch maple cubes. Each cube is to correspond to a prime number according to the following:

	Prime	Quantity Required	Quantity To Produce
1	2	97	107
2	3	48	53
3	5	24	26
4	7	16	18
5	11	9	10
6	13	7	8
7	17	5	6
8	19	5	6
9	23	4	5
10	29	3	4
11	31	3	4
12	37	2	3
13	41	2	3
14	43	2	3
15	47	2	3
16	53	1	2
17	59	1	2
18	61	1	2
19	67	1	2
20	71	1	2
21	73	1	2
22	79	1	2
23	83	1	2
24	89	1	2
25	97	1	2
Totals		**239**	**279**

- Note that the quantity to produce is larger than the quantity required. This allows students the opportunity to make “mistaken” prime factorizations while constructing and it provides insurance against lost or misplaced blocks. This list and a link to a maple block supplier is supplied in an Excel document found at this link.

- Each block corresponding to a prime number must be painted the same color. For example, 107 blocks corresponding to the prime number 2 should all be painted blue, 53 blocks corresponding to the number 3 should all be painted yellow, and so on. Acrylic paint applied with a foam brush works best.
- Each block corresponding to a prime number should have that prime number written around the “waist” of the block on four faces. The top of the block should be given the multiplication “dot” and the bottom of the block should be blank. This arrangement gives each block a single right-side-up orientation and reduces confusion associated with handwritten numbers (i.e. an upside-down 2 looks like a 5). Paint pens work good for this, one black for light colored blocks, one white for dark colored blocks. A picture of painting supplies previously used is one jpg file found at this link.

Handouts

- You will need the six handouts that give number assignments to small groups. These handouts are labeled: Group A, Group B, Group C, Group D, Group E, Group F. These handouts are available at the end of this document.

Lesson Plan

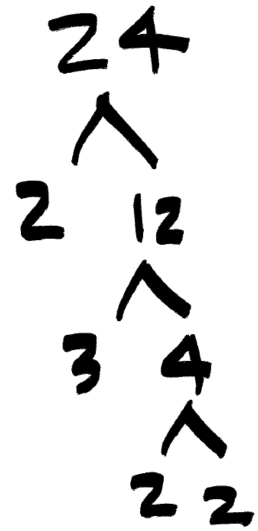
Small Group Assignment

- Students need to be randomly placed in six small groups of 3-4 students per group

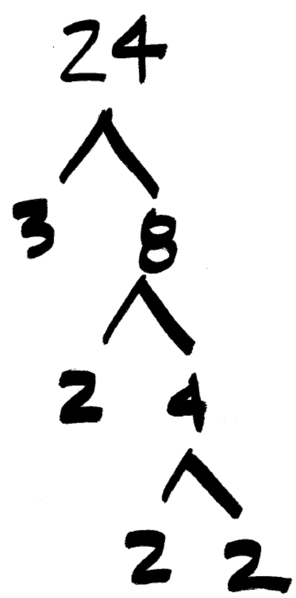
Review of Prime Numbers, Composite Numbers and Prime Decomposition

- Begin by reviewing prime numbers
  - Who can tell me what a prime number is?
    Answer: A prime number is any positive integer greater than one that has exactly two distinct divisors, one and itself. Examples: 2, 3, 7, 11, 13
  - Turn to a neighbor and give them one example of a prime number.
  - Who can tell me what a composite number is?
    Answer: A composite number is any positive integer greater than one that is not prime. A composite number is any positive integer greater than one that has more than two divisors. Examples: 4, 6, 9, 12, 18
  - Turn to a neighbor and give them one example of a composite number.
  - What about 1, is 1 prime or composite?
    Answer: Students will, most likely, argue that 1 is prime as it is divisible by "one and itself" just like 2, 3, 7, 11 and 13. Take time to address this common misconception by asking the follow question, "How is 1 different from the prime numbers 2, 3, 7, 11 and 13?" Or, maybe more directly, "How many divisors does 1 possess?" followed by, "How many divisors does every prime number possess?" Since 1 has only one divisor, it is not prime because it does not have "exactly two distinct divisors". Since 1 has only one divisor, it is not composite because it does not have "more than two divisors".
- Review the prime numbers found in 2-100. A poster found at this link works well with post-it notes to enact the Sieve of Eratosthenes to find these primes in a preceding lesson. The completed poster should remain in the classroom for reference throughout this activity.

- Review prime decomposition
  - How can we use a factor tree to write a number, say 24, as a product of primes?
    Possible Answer:



  - If we started the factor tree differently, would we get a different prime factorization for 24? Let's check.
    Possible Answer:



  - Does every number have only one prime factorization? Does order matter?
    Answer: Based on the analysis of 24=2X2X2X3 it appears that numbers have a unique prime factorization, proof of this conjecture is difficult, and so, we assume it true until later study.

What does a number's prime factorization tell us?

- Motivation
  - A number's prime factorization is an important representation that helps us understand a number in a multiplication sense. Primes are the building blocks for all numbers similar to how the atomic elements are the building blocks for all molecules. We study the primes, as we study the elements, because they are important predictors.
  - In order to understand how primes are predictive, I want you to help me build what we are going to call "the city of numbers".
  - In the city of numbers, every number 2-100 is represented by a tower of primes. This tower is a representation of each number's prime factorization.
  - Let's take the example of 24 again, what tower should I place on the space labeled 24?

    Answer: three "two" blocks and one "three" block.
- Group Work
  - Now I want you to help me build the rest of the towers. Each group is going to be assigned 8 numbers to investigate. (Pass out the group number assignments, see handouts at the end of this lesson).
  - Find the prime factorization for each of the eight numbers. Write these factorizations in the box for that number
  - Once you think you have found all eight prime factorizations, build each number as a tower of primes. Place each tower in the box for that number on the handout.
- Teacher Notes on Group Work
  - Some groups will be faster than others to find all 8 prime factorizations
  - For groups that are faster, ask them to help other groups finish
  - Do not proceed until all groups have found all 8 prime factorizations
- Analyzing Towers – Seeing Divisibility
  - Look at the eight towers you have constructed, how can you tell if one of the numbers is divisible by 3 by looking at the tower?

A	B	C	D	E	F
100	98	94	92	99	86
96	87	90	91	95	77
93	84	76	85	88	75
82	80	69	78	66	72
81	70	65	74	64	62
68	56	58	63	51	54
60	55	57	46	45	42
52	30	20	15	12	10

Answer: The tower has a 3 in it.

- Look at the eight towers you have constructed, how can you tell if one of the numbers is divisible by 6 by looking at the tower?

A	B	C	D	E	F
100	98	94	92	99	86
96	87	90	91	95	77
93	84	76	85	88	75
82	80	69	78	66	72
81	70	65	74	64	62
68	56	58	63	51	54
60	55	57	46	45	42
52	30	20	15	12	10

  Answer: The tower has a 3 and a 2 in it.

- Look at the eight towers you have constructed, how can you tell if one of the numbers is divisible by 15 by looking at the tower?

A	B	C	D	E	F
100	98	94	92	99	86
96	87	90	91	95	77
93	84	76	85	88	75
82	80	69	78	66	72
81	70	65	74	64	62
68	56	58	63	51	54
60	55	57	46	45	42
52	30	20	15	12	10

  Answer: The tower has a 3 and a 5 in it.

- Analyzing Towers – Seeing Divisibility the Case of 60
  - Now that you have had some time to think about what a number's prime factorization tells us about which numbers do and do not divide it, let's take a look at the numbers that divide 60. Each group has a single number that is a divisor of 60 in their group of 8 numbers. Please bring that number up to the grid and place it on the correct location.
  - What you notice about the prime factorization of 60 and the prime factorization of the divisors of 60?
    Answer: All factors of 60 are some subset of the prime factorization of 60 = 2X2X3X5. For example, 30 is a factor of 60 and 30 = 2X3X5.

- Discovery Learning – Building All Towers 2-100
  - Now that you have had some time to think about divisors of 60, let's expand our view. Please carefully place the rest of your towers on the grid. Be careful not to knock over others towers.
  - Now that we have placed all our towers there still are some that need to be constructed. Choose a missing tower location and build and place its tower.
  - Now that we have the completed tower of primes landscape, let's take some time to look for patterns. Let's take one minute to look quietly at this amazing representation of numbers, then, we will take one minute to share what we see with a partner and then we will share out what we have discovered with the large group.

    Many patterns are possible, some patterns students have observed include:
    - 2s are only found in the even columns
    - 5s are only found in the 5^{th} and 10^{th} column
    - Every number in the 10^{th} column has both a 2 and a 5. Nowhere else are both a 2 and a 5 found together.
    - 11s stair step in a diagonal fashion starting at 11 and moving up one row and over one column (resulting in the addition of 10 and 1, or 11)
    - There are many diagonal patterns involving 3s, for instance, if you start at any multiple of 3 another multiple of 3 can be found by moving up one row and over two columns (resulting in the addition of 10 and 2, or 12), so, adding 12 to a multiple of 3 yields another multiple of 3.
    - Another commonly noticed pattern involving 3 is the observance that any time there are two 3s you can find another number with two 3s by moving up one row and back one column (resulting in the addition of 10 and -1, or 9)
    - Starting with any multiple of 7 we can find another multiple of 7 by moving up two rows (+20) and over one column (+1) – so adding 21 to any multiple of 7 yields another multiple of 7. This rule also works for 3.
    - Almost all of the primes are found in the 1^{st}, 3^{rd}, 7^{th} and 9^{th} columns…this makes sense because any two digit number ending in 0, 2, 4, 5, 6, or 8 has at least one divisor other than one and itself
    - The tallest towers tend to include many small primes (i.e. 2s or 3s)
    - Every other multiple of 4 has three 2s (i.e. the even multiples of 4 are also multiples of 8)

- More Practice
  - Let's move on to practicing what we have learned. Please complete practice problems 1 and 2. If you are up for a bigger challenge, try the last problems numbers 3 and 4. What you do not finish in class, please complete as homework for tomorrow.

Handouts – Group Number Assignments

Group A

100	81
96	68
93	60
82	52

Group B

98	70
87	56
84	55
80	30

Group C

94	65
90	58
76	57
69	20

Group D

92	74
91	63
85	46
78	15

Group E

99	64
95	51
88	45
66	12

Group F

86	62
77	54
75	42
72	10

Handouts – Practice Problems

Practice Problems **Name:________________________**

1. The prime factorization of a number N is N=2X2X2X3X3. Which of the following numbers are divisors of the number N? Explain your decision for each.

    a. 3

    b. 5

    c. 16

    d. 8

    e. 18

    f. 20

    g. 27

2. The prime factorization of the number 56 is 2X2X2X7. Use the prime factorization to find all of the divisors of 56. Show how you found them.

3. **Extra Challenge:** Suppose that a number is divisible is by both 14 and 16. What is the smallest number it could be? Explain your reasoning.

4. **Extra Challenge:** Suppose that a number is divisible by each of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. What is the smallest number it could be? Explain your answer.

Tower Key

91 $7\cdot13$	92 $2^2\cdot23$	93 $3\cdot31$	94 $2\cdot47$	95 $5\cdot19$	96 $2^5\cdot3$	97 97	98 $2\cdot7^2$	99 $3^2\cdot11$	100 $2^2\cdot5^2$
81 3^4	82 $2\cdot41$	83 83	84 $2^2\cdot3\cdot7$	85 $5\cdot17$	86 $2\cdot43$	87 $3\cdot29$	88 $2^3\cdot11$	89 89	90 $2\cdot3^2\cdot5$
71 71	72 $2^3\cdot3^2$	73 73	74 $2\cdot37$	75 $3\cdot5^2$	76 $2^2\cdot19$	77 $7\cdot11$	78 $2\cdot3\cdot13$	79 79	80 $2^4\cdot5$
61 61	62 $2\cdot31$	63 $3^2\cdot7$	64 2^6	65 $5\cdot13$	66 $2\cdot3\cdot11$	67 67	68 $2^2\cdot17$	69 $3\cdot23$	70 $2\cdot5\cdot7$
51 $3\cdot17$	52 $2^2\cdot13$	53 53	54 $2\cdot3^3$	55 $5\cdot11$	56 $2^3\cdot7$	57 $3\cdot19$	58 $2\cdot29$	59 59	60 $2^2\cdot3\cdot5$
41 41	42 $2\cdot3\cdot7$	43 43	44 $2^2\cdot11$	45 $3^2\cdot5$	46 $2\cdot23$	47 47	48 $2^4\cdot3$	49 7^2	50 $2\cdot5^2$
31 31	32 2^5	33 $3\cdot11$	34 $2\cdot17$	35 $5\cdot7$	36 $2^2\cdot3^2$	37 37	38 $2\cdot19$	39 $3\cdot13$	40 $2^3\cdot5$
21 $3\cdot7$	22 $2\cdot11$	23 23	24 $2^3\cdot3$	25 5^2	26 $2\cdot13$	27 3^3	28 $2^2\cdot7$	29 29	30 $2\cdot3\cdot5$
11 11	12 $2^2\cdot3$	13 13	14 $2\cdot7$	15 $3\cdot5$	16 2^4	17 17	18 $2\cdot3^2$	19 19	20 $2^2\cdot5$
1 1	2 2	3 3	4 2^2	5 5	6 $2\cdot3$	7 7	8 2^3	9 3^2	10 $2\cdot5$

Acknowledgements

This lesson is informed by two sources:

Sinclair, N., Zazkis, R., & Liljedahl, P. (2004). Number worlds: Visual and experimental access to elementary number theory concepts. *International Journal of Computers for Mathematical Learning*, 8, 235-263.

Burkhart, J. (2009). Building numbers from primes. *Mathematics Teaching in the Middle School*, 15(3), 156-167.

Sinclair et al.'s article was the genesis for a "gridded representation of number" where multiplicative structure can be explored. Burkhart uses blocks for primes. I combined these two ideas to arrive at my innovation – a gridded representation of 1-100 in both base-ten and prime-factored-form where towers of prime blocks represent each number's prime decomposition. I thank these authors for these ideas.