MOMATH

NATIONAL MUSEUM OF MATHEMATICS

THE 2018 ROSENTHAL PRIZE

for Innovation in Math Teaching

Squareland: Into How Many Squares Can We Cut a Square?

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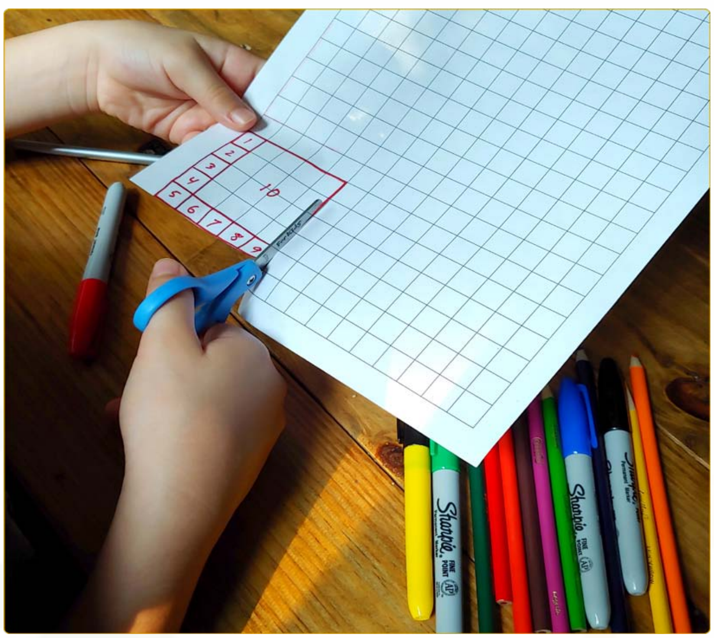

Lesson Plan: Adaptable for Grades 5-8

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Into How Many Squares Can We Cut a Square?

Lesson Goals

To generate and analyze patterns while exploring what it means to conjecture and prove mathematical claims.

The primary goal of every lesson I teach is to enable learners to become creators of knowledge and critical thinkers. To advance this vision, it is vital to provide a relaxed atmosphere where every student can enjoy the journey of learning through meaningful explorations. This ethos permeates this activity.

I want students, from elementary school to college, to conjecture and prove as the main aspect of their mathematical experience. This will help them to appreciate mathematics, to quote Mark Saul, "as the art of figuring things out."

As students conjecture or look for counterexamples, they exercise and hone their critical-thinking skills. Moreover, the geometric aspect of this problem allows students to visualize the proofs of the conjectures they make.

Student Outcomes

Students generate and analyze patterns as they physically or mentally cut squares into smaller squares. As they go along, they conjecture and try to prove their claims. For some cases, they put forth impossibility arguments. In terms of differentiation, all students understand the main claim arising from this problem (below), some understand why certain cases do not work, while those performing at the highest levels can prove the main claim rigorously:

Let n be a natural number. If we cut a square into n pieces such that every resulting piece is also a square, then this can be done for every n, except for $n = 2, 3,$ and 5.

These outcomes meet or exceed all eight Common Core Standards for Mathematical Practice, and stretch the boundaries of the Standards for Mathematical Content insofar as I incorporate mathematical rigor through conjectures and proofs—the great missing elements in most math classrooms.

Common Core Standards for Mathematical Practice

This lesson *squarely* hits all eight Standards for Mathematical Practice. Indeed, when we select meaningful mathematics activities, it is difficult not to address all of them.

CCSS.MATH.PRACTICE.MP1 *Make sense of problems and persevere in solving them.*

CCSS.MATH.PRACTICE.MP2 *Reason abstractly and quantitatively.*

CCSS.MATH.PRACTICE.MP3 *Construct viable arguments and critique the reasoning of others.*

CCSS.MATH.PRACTICE.MP4 *Model with mathematics.*

CCSS.MATH.PRACTICE.MP5 *Use appropriate tools strategically.*

CCSS.MATH.PRACTICE.MP6 *Attend to precision.*

CCSS.MATH.PRACTICE.MP7 *Look for and make use of structure.*

CCSS.MATH.PRACTICE.MP8 *Look for and express regularity in repeated reasoning.*

Common Core Standards for Mathematical Content

CCSS.MATH.CONTENT.5.OA.B.3 *Analyze patterns and relationships.*

CCSS.MATH.CONTENT.6.EE.A.2 *Write, read, and evaluate expressions in which letters stand for numbers.*

CCSS.MATH.CONTENT.7.EE.B.4 *Solve real-life and mathematical problems using numerical and algebraic expressions and equations.*

CCSS.MATH.CONTENT.7.G.B.6 *Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.*

To develop a lesson, the usual procedure recommended by math specialists is to choose the content standard first, and then choose an activity to support that particular standard. I prefer to select mathematically meaningful activities first, and then look for standards to justify my use of them in the classroom.

Prerequisite Knowledge

As Euclid will tell you, a square is a rectangle the sides of which are all congruent. A rectangle is a parallelogram with right interior angles.

Time Required

Preparation time: If you understand the problem, 15 minutes. If you have never seen the problem, 60 minutes.

Class time: 45-90 minutes, depending on how much depth the teacher wants.

Materials

Paper, pencils, graph paper, and scissors (Optional: colored pencils and enlarged graph paper printed on color paper. See Appendix 1.)

Essential Questions

Can I cut a square into any given number of squares such that every piece of the original square is also a square? If I can, how do I do it? If I cannot, why not?

Activity

Activating question [3 min]: *Into how many squares can we cut a square?*

Have students discuss this question among themselves for two to three minutes.

Questions immediately arise:

> *What do you mean by "cut"?*
> *Do squares need to be the same size?*
> *Must the sides of the pieces be parallel to the sides of the original square?*

Discovery [7 min]: *Can you cut a paper square into 11 squares so that every piece is a square?*

The purpose of this question is to refine and give context to the opening question. With that in mind, let students explore it for a few minutes with minimal assistance. Then suggest using graph paper, especially if you want the activity to move faster. Using graph paper will help students correct some common misconceptions about rectangles and squares. For example, if they start drawing squares on a blank piece of paper, many will cut it into a combination of rectangles of different sizes, only some of which are squares. That misconception springs from a lack of understanding of what a square is. In other words, it points to a lack of understanding of the definition of a square, but given graph paper, students generally do not make this error. Nevertheless, we can use this teaching moment to emphasize the importance of definitions in mathematics, and in fact, in any logical discussion.

Exploration [20 min]: *For what natural numbers can we cut a square into that many pieces? How about 1, 2, 3, 4, 5, 6? Are there any obvious ones?*

By this point, students will hit upon many different cases. You may want to ask the following questions in order to elicit from students the idea of organizing their findings in a chart. Notice that the higher you start on this tiered questioning, the more you are demanding from students. I would try to avoid simply telling them to use a chart (although I have done just that!). With this approach, I am trying to encourage logical and creative reasoning throughout the lesson.

Does anyone have an idea about what to do with all of these cases?
Do we organize them?
How do we organize them?
Would it be useful to organize them via a chart?

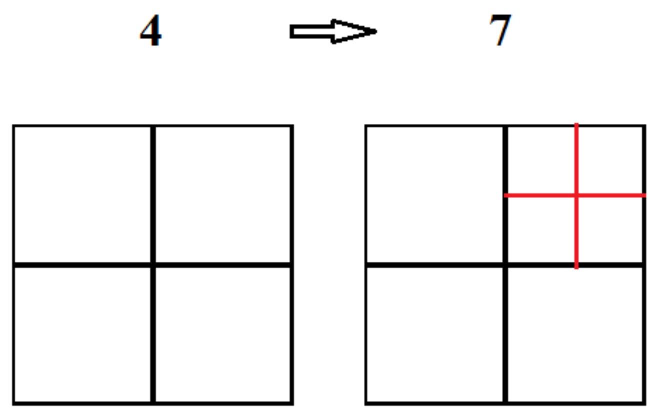
At this point, students are organizing their data collectively, in small groups, or individually. (Though I encourage collaborative work with this activity, I also let students work in whatever way they feel comfortable, as long as I provide appropriate support for each circumstance.) Consider having a student keep track on the board. You may also use the handout in Appendix 2.

Number of Squares	Yes or No
1	Yes
2	No
3	No
4	Yes
5	No
6	Yes
7	Yes
8	Yes
9	Yes
10	Yes
11	Yes
12	Yes
13	Yes
14	Yes
15	Yes
16	Yes

Completing the chart above can be quite challenging. In my experience, 8 and 11 are some of the hardest cases. We will later show why you cannot do 2, 3 or 5, but you might want to have fun trying to find a way to explain to children why this is true before you read the details.

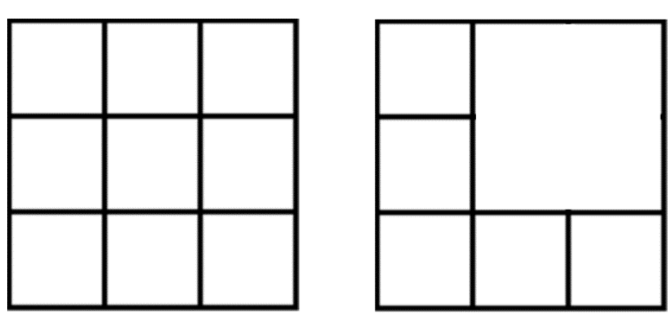
Deepening the Exploration [10 min]: *Can you cut a square into 29 squares? 30? Do you have a way to do it?*

By now, students will have noticed that every time you cut a square into four equal squares (with what one student in my recorded lesson called a "plus cut") you add three squares to the total number of squares you had.

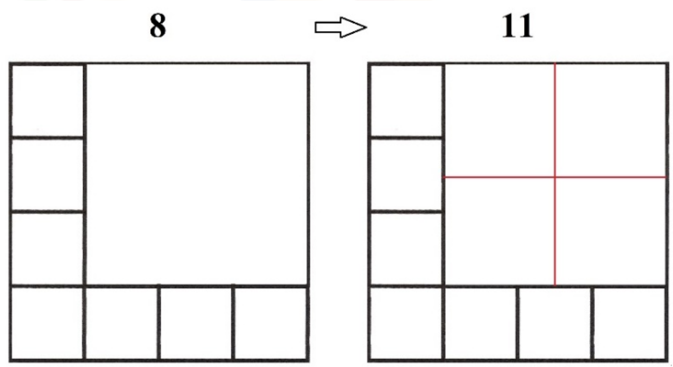


Hence, 4 leads to 7, which leads to 10, and so on.
Likewise, 6 leads to 9, which leads to 12, and so on.
Finally, 8 leads to 11, which leads to 14, and so on.

This, of course, assumes that students were able to find 6 and 8. Since 6 is within the sequence containing 9, students might at first argue (correctly) that they have a way to generate 12, 15, and so on. In fact, if they reverse that technique, they can generate 6 from 9.



The previous process usually leads them to finding 8 and 11.



We now have the following sequences—starting from 6, 7, and 8—to generate every natural number starting from 6.

6, 9, 12, 15, …
7, 10, 13, 16, …
8, 11, 14, 17, …

Some students now quickly realize that to answer the question of how to cut a square into 29 squares, it suffices to find the list in which 29 falls. Some may actually try all sequences, and realize it is the one starting with 8. Encourage them to make connections to their knowledge of divisibility. In that pursuit, you may ask:

Does this remind you of something we have seen?

Once they have made the connection to remainders, help them generate the following chart. You may either do the following chart on the board, or provide a handout where they keep track of their findings.[1] The algebraic representation shown in the middle column is an excellent connection to material usually introduced at the beginning of middle school, consistent with CCSS 6.EE.A.2 on page 2. It can also serve as an introduction to modular arithmetic. For this case, $n \in \mathbb{N}$ with $n \geq 2$.

Sequence	Algebraic Representation	Remainder when divided by 3
6, 9, 12, 15, …	$3n$	0
7, 10, 13, 16, …	$3n+1$	1
8, 11, 14, 17, …	$3n+2$	2

That is, all we need is to find the remainder when dividing the number by 3 and we are done! (Well, almost done. We still need to deal with 2, 3, and 5. Use this as an exit ticket after summarizing.)

Summarizing [4 min]: Can you cut a square into 2018 squares? How about 2019?

[1] See Appendix 2.

Exit Ticket [1 min]: "Why can we not cut a square into 2, 3 or 5 squares?" (Ideally, you will explore the question the next time class meets.)

Questioning Guide Tips

I have incorporated questions throughout this lesson plan to guide other teachers as they implement this lesson. In general, always start your questioning with the intent of probing students' understanding of the problem. As obvious as it may seem, experience tells me this is often the missing (and ignored) link in questioning for understanding, even when students seem engaged scribbling or doodling.

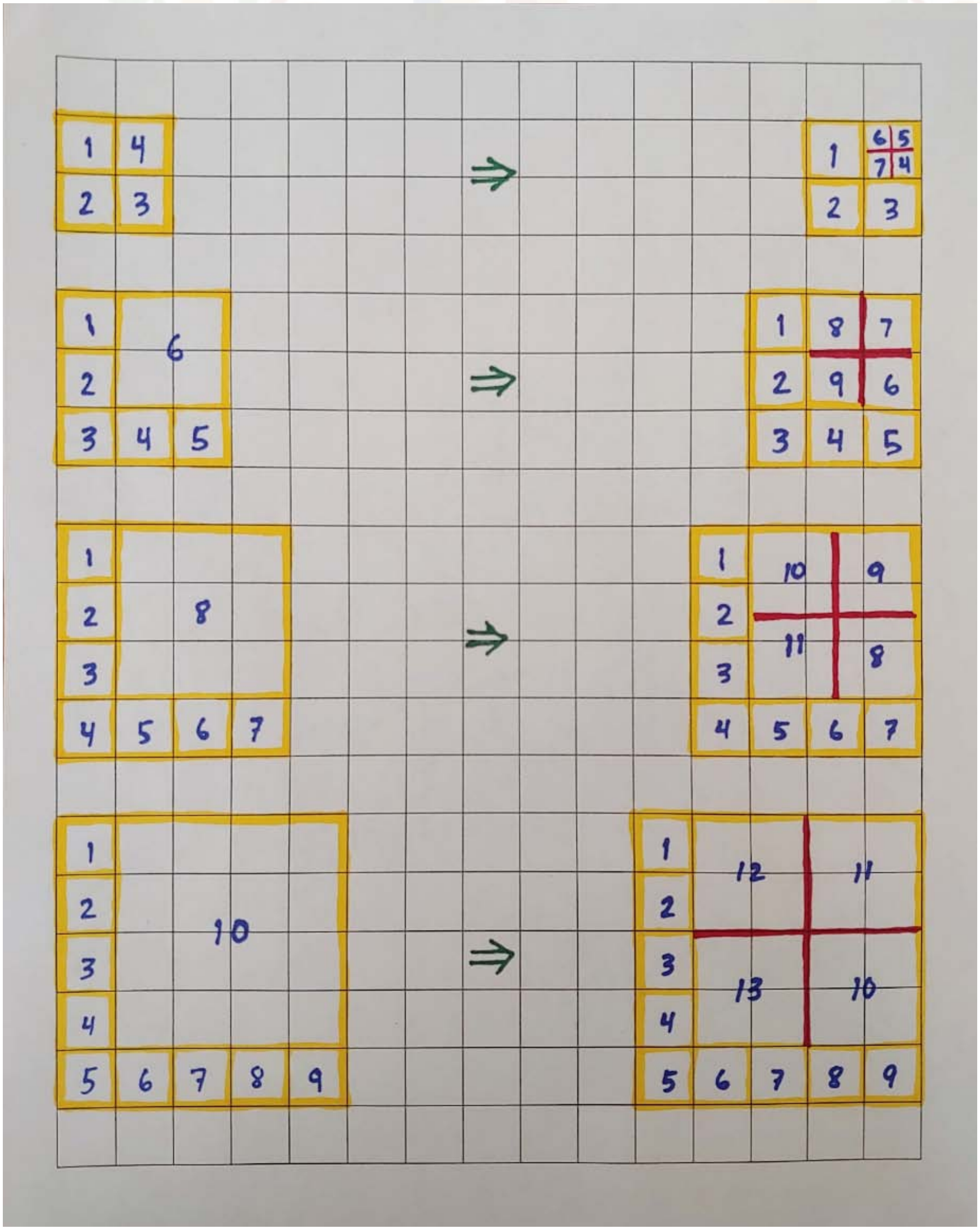
Once you have verified that students understand the problem, avoid giving hints about how to solve it until many are on the verge of frustration (or already frustrated). Productive struggle is important. In short, give a hint only after it is clear that students understand the problem, but are stuck. A hint given at the wrong time, either too soon or too late, can ruin the creative process for everyone in the room.

Adaptations and Extensions

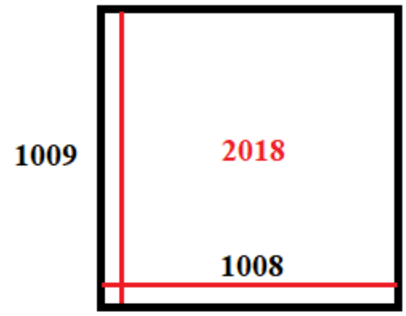
We are now ready for a stronger conjecture: *We can cut a square into any number of squares, except for 2, 3, and 5. Can you prove it?*

As much as I like the solution presented in this lesson, I prefer the following one because of its beauty and simplicity.

For this approach (see the diagram on the next page), we will use the technique we applied to find 11 from 8, and notice that from any even number n, we can generate the odd number $n + 3$. We now have just two sets—even and odd—that are connected in a simple and elegant way. Notice also that the side of the square with which we start can be conveniently set at $\frac{n}{2}$. For example, we cut the 4x4 square below into 8 squares, and $4 = \frac{8}{2}$.



Hence, if we want to find 2018, simply start out with a square of side 1009, cutting the 1009+1008 unit squares along two adjacent edges, and adding the larger square (of side 1008) to complete our 2018 squares.

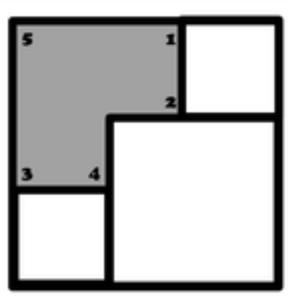


To do 2019, start with a square of side 2019 – 3 = 2016, and proceed in a similar fashion as above.

I clarify that before selecting this problem for this contest, I was not aware that this problem had appeared—thanks to Daniel Finkel—on *The New York Times*' column *Numberplay* on April 4, 2013 in the garb of squareable numbers. That was one year after I had started using this problem with the approach presented here. Moreover, in one of its puzzle booklets, the Julia Robinson Mathematics Festival presents Finkel's version.

Since for this extension we still need to prove that we cannot cut a square into 2, 3, or 5 squares, I will defer to Finkel.

> *Finally, we have to prove that 2, 3, and 5 are not squareable. There has to be a square in each corner of the large square, and since no two corners can be filled with the same smaller square (unless you have just a single square), you need at least 4 squares. So 2 and 3 are not squareable.*
>
> *The tricky one is 5. Suppose you're combining five squares to form a single square with side length 1 unit. You can't have all the smaller squares with side length less than or equal to 1/2, since this wouldn't cover every side of the unit square, or you would have used exactly four squares. So one square must have side length greater than 1/2.*



> *Adding in the squares to cover the adjacent corners to that square, we must have a picture like this.*

But this means we need to fill in a symmetrical, non-convex hexagon with precisely two squares. This is impossible, because you need a square to cover angles 1 and 2, and a square of the same size, by symmetry, to fill in angles 3 and 4. Hence, either they both cover angle 5, or neither does. Either way, this construction fails. Hence, 5 is not squareable.

Future Explorations: Cubes and Polyominoes

Cube Dissection

According to J. Mason, L. Burton, K. Stacey in *Thinking Mathematically*, published in 2010, there is a 3-dimensional version of this problem that is open. To paraphrase Mason et al. using language consistent with what I have used in this lesson plan, we can pose the problem as follows:

Into how many cubes can we cut a cube so that every piece is a cube?

They call the relevant numbers "very nice" and state (page 86), "It is conjectured that all numbers bigger than 47 are 'very nice' but little is known at present." However, this is incorrect.[2]

Before I became aware of the resolution to this problem, I was able to prove that all integers greater than 54 are "very nice."[3] I am confident that your most capable students will enjoy playing with this claim and trying to find ways to represent some smaller cases with manipulatives. I recommend using Rubik's cubes of different sizes (in particular, 2x2x2, 3x3x3, 4x4x4, 5x5x5, and 6x6x6) to help them visualize cuttings.

This extension is great to nurture students' spatial abilities, consistent with CCSS 7.G.B.6 on page 2. You can also invite students to explore an extension in hyperspace by first showing them *Flatland: The Movie*, with the voices of Martin Sheen and Kristen Bell.[4] I might try that next![5]

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[2] The Rosenthal review committee recommended this reference about two weeks before my final submission. I tried to prove it as soon as I read it. However, I then found that the problem had been posed in 1946 by Hugo Hadwiger and solved in 1977 independently by D. Rychener and A. Zbinden. (See http://mathworld.wolfram.com/HadwigerProblem.html.)

[3] See Appendix 3 for a sketch of my proof.

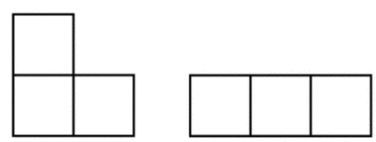
[4] In the interests of full disclosure, the consultants for *Flatland: The Movie* were the principals of Hollywood Math and Science Film Consulting, whom I too have worked with. The filmmakers say that over a million schoolkids have watched *Flatland: The Movie*.

[5] Amusingly, National Public Radio has called Jason Zimba "the man behind Common Core Math." His doctoral advisor at Oxford University was Roger Penrose, famous for his non-periodic tilings of the plane. According to Jonathan Farley, also a graduate student at Oxford at the time, Penrose made a bet with fellow Oxford professor David Deutsch over a puzzle, the solution to which one of them put in a sealed envelope. If the other was able to solve it, he'd get one pound: You have a rectangle that you want to tile with smaller rectangles. For each small

Polyominoes

I like to follow up with problems involving *polyominoes*. Polyominoes are two-dimensional shapes formed by adjoining squares edge to edge. More specifically, if they consist of two squares, we call them dominoes; three squares, trominoes; four squares, tetrominoes; five squares, pentominoes; *n* squares, *n*-ominoes. We consider them equivalent if we can obtain one from another by translation, rotation, or reflection. If we allow reflections, we call them free polyominoes; if we do not, we call them fixed polyominoes. Martin Gardner first used the term *super-dominoes*, but the term coined by Solomon Golomb (polyominoes) has prevailed.

Begin by defining polyominoes visually. I like to bring square tiles of the same size and different colors. Then show the two trominoes (3-ominoes). Ask students to determine the total number of distinct free and fixed trominoes. There are only two, as you can easily verify. (It is good to observe that the two free trominoes are identical to the two fixed trominoes because we can get the 'flip' of the tromino at the left in the picture below by rotating it 180 degrees without flipping.



Trominoes

Do the same with tetrominoes (4-ominoes), and free pentominoes (5-ominoes).

Before you do the pentominoes, you may want to explore the following question with your students.

Can you cover a 4x5 board with all five tetrominoes (one of each)?

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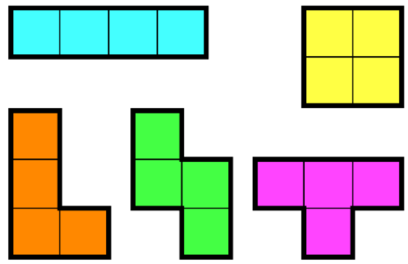
rectangle, either the length is an integer or the width is. Prove that for the big rectangle, either its length is an integer or its width is an integer.

Also amusingly, Roger Penrose successfully sued Kleenex for using "his" aperiodic tilings in a design, but in fact such aperiodic tilings can be found in religious buildings in Spain that are centuries old. This underscores the value of both diversity and history in the teaching of mathematics (even to future lawyers).



Tetrominoes

Most students will immediately say that they can do it because $4 \times 5 = 20$, and the five pentominoes collectively have 20 squares. Prompt them to show you how they would cover the 4x5 board. After a while, they will realize that something is wrong with their reasoning. This is an excellent and beautiful way to introduce coloring techniques and the pigeonhole principle. I have used this problem successfully with elementary school children in math circles' sessions.

Appendix 1: 15x15 Grid

Appendix 2: Handouts

Number of Squares	Yes or No

Sequence	Algebraic Representation	Remainder When Divided by 3

Appendix 3: Devil's Cube[6]

Claim: *We can cut a cube into n pieces such that every piece is a cube, for* $n \geq 48$.

Sketch of proof: Since the 6x6x6 cube holds the key to a solution to this problem, I call it the Devil's Cube ("666").

We will start with the Devil's Cube and cut out 3x3x3 cubes to generate all cases. (See the table below.) It is the smallest cube that allows for a lower bound.

0. Start with a 6x6x6 cube and cut 0 3x3x3 cubes. That yields 6 mod 7.
1. Start with a 6x6x6 cube and cut 1 3x3x3 cube. That yields 1 mod 7.
2. Start with a 6x6x6 cube and cut 2 3x3x3 cubes. That yields 3 mod 7.
3. Start with a 6x6x6 cube and cut 3 3x3x3 cubes. That yields 5 mod 7.
4. Start with a 6x6x6 cube and cut 4 3x3x3 cubes. That yields 0 mod 7.
5. Start with a 6x6x6 cube and cut 5 3x3x3 cubes. That yields 2 mod 7.
6. Start with a 6x6x6 cube and cut 6 3x3x3 cubes. That yields 4 mod 7.

Number of 3x3x3 cubes cut out of a 6x6x6 cube	Number of cubes generated, *a*	*a* mod 7
0	216	6
1	190	1

[6] I thank Jonathan Farley, a Harvard- and Oxford-trained mathematician, for his revision of this argument. Farley has made a career of solving long-standing open problems, most notably, problems posed by MIT's Richard Stanley, Richard Rado, E.T. Schmidt, George Grätzer, Bjarni Jónsson, and UC Berkeley's Ralph McKenzie. His input was invaluable.

2	164	3
3	138	5
4	112	0
5	86	2
6	60	4

To reduce the number of cubes in each case, simply subtract as many 2x2x2 cubes as possible after cutting the initial number of 3x3x3 cubes. This will not alter the modularity since the way we generate more cubes in the sequence is precisely by taking a cube and slicing it into eight cubes (8-1=7).

Moreover, if we start with smaller cubes and carry out a similar process, we obtain the lower bounds needed to settle our claim. In the chart below, we include the pertinent cuttings. For instance, the last row indicates that we start with a Devil's Cube (216 unit cubes), subtract six 3x3x3 cubes (6x27=162 unit cubes), subtract two 2x2x2 cubes (16 unit cubes), and lastly add one cube for every cube we have cut out (in this case, 8). That leaves 46 cubes. Now take one cube, *any* cube, and cut it into eight cubes to obtain $46 - 1 + 8 = 53$ cubes. Continue this process *ad infinitum*, that is, take one cube and cut it into eight cubes to generate another possible number. Since the remainder when we divide 46 by 7 is 4, we can generate the class 4 mod 7 starting at 46. A full proof might require illustrations showing that the last four cases are indeed attainable. The arithmetic alone is no guarantee that the geometry will allow it. I actually drew the cubes in question to ascertain my claim. For instance, consider 5 mod 7 in the penultimate row. Although the arithmetic will allow for 216-3*27-13*8+16 = 47, apparently the geometry will not (it seems you can cut out at most eleven 2x2x2 cubes after cutting out three 3x3x3 cubes).[7]

Side-length	Cuttings	First term of sequence, *a*	*a* mod 7	Sequence
1		1	1	1, 8, 15, 22, 29, 36, 43, 50, 57, ...
3	27-8+1	20	6	20, 27, 34, 41, 48, 55, ...
4	64-27+1	38	3	38, 45, 52, 59, ...
6	216-3*27-11*8+14	61	5	61, 68, 75, ...
6	216-5*27-5*8+10	51	2	51, 58, 65, ...
6	216-6*27-2*8+8	46	4	46, 53, 60, ...
8	512-6*64-2*27-4*8+12	54	5	54, 61, 68, ...

[7] The case with side-length 8 is due to D. Rychener and A. Zbinden. It is the only case that eluded me. Remember, however, that this problem took 30 years to be settled. I pleased with my independent progress on this beautiful problem.

Combining cases: 1,8,15,20,22,27,29,34,36,38,41,43,45,46,48,49,50,51...

QED

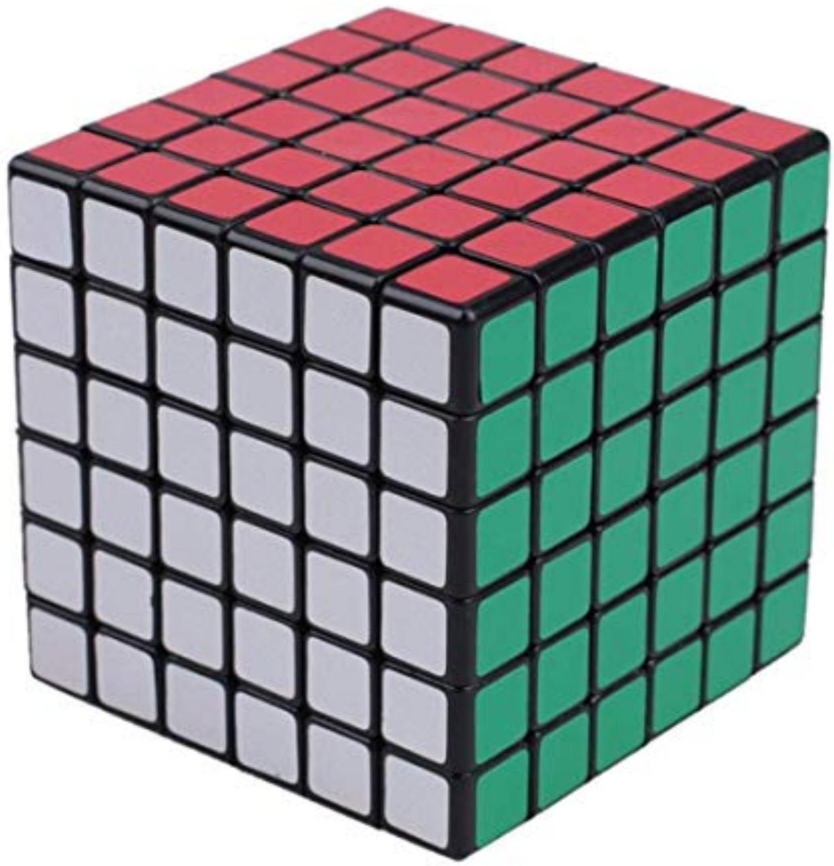
Conjecture 1: *All of the following cases are unattainable:*

2-7,9-14,16-19,21,23-26,28,30-33,35,37,39-40,42,44, and 47.

Conjecture 2: For any dimension of hypercube, there will be at most finitely many unattainable numbers of pieces into which we can cut our hypercube. The highest unattainable number increases along with the dimension. It might be fruitful to consider the asymptotics of that number (5 for the two-dimensional case, 47 conjecturally for the three-dimensional case).

Conjecture 3: For any dimension of hypercube, the *number* of unattainable numbers increases along with the dimension. It might be fruitful to consider the asymptotics of that number (3 for the two-dimensional case, 33 conjecturally for the three-dimensional case). We can ask students to ponder the connection between Conjectures 2 and 3.

N.B. My 11-year-old daughter Rohini actually worked with me on this problem. The fact that she understood the technique I employed leads me to believe that other children her age will enjoy and understand this beautiful problem.



The Devil's Cube (photo from Amazon.com)