NATIONAL MUSEUM OF MATHEMATICS

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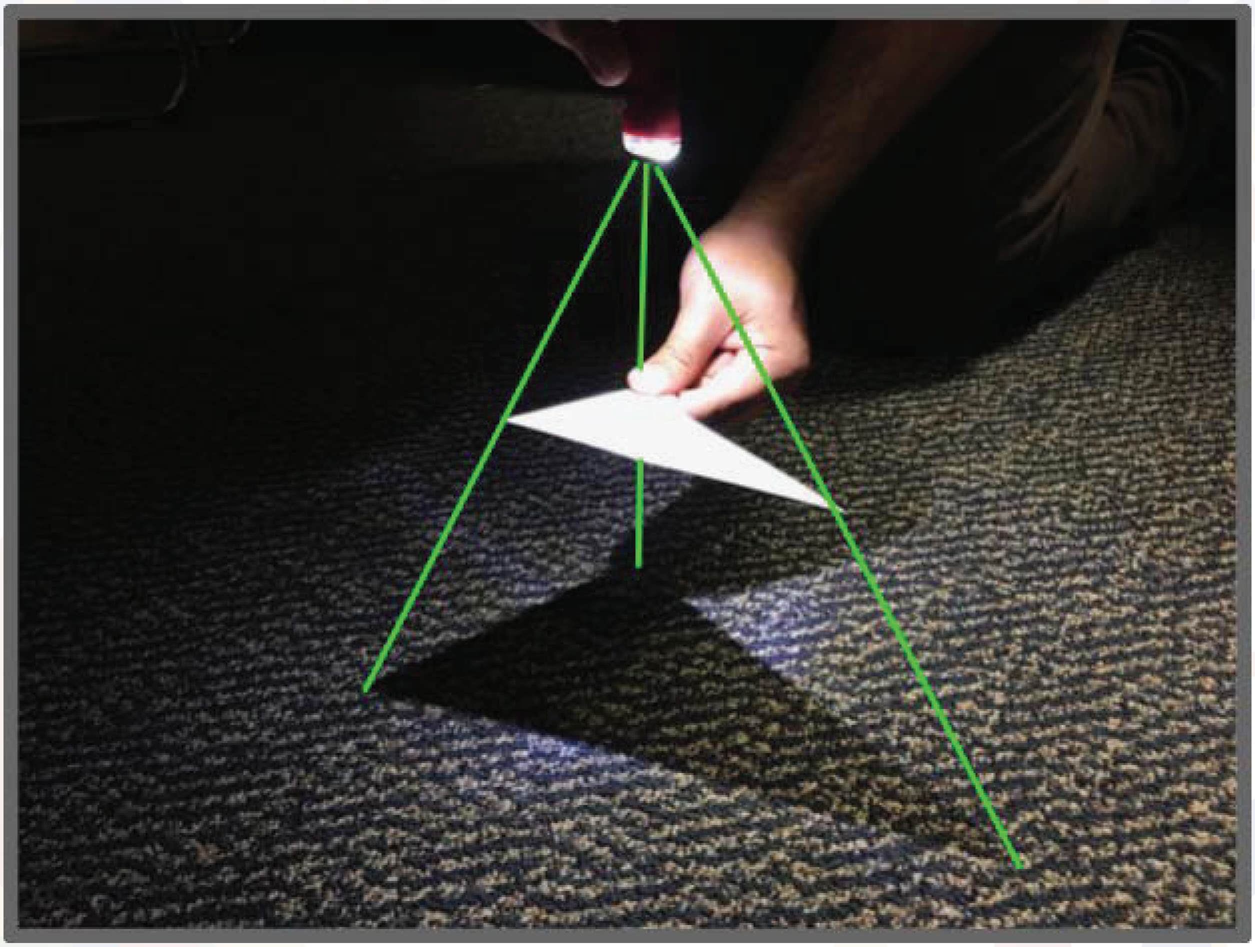
for Innovation in Math Teaching

WINNING ACTIVITY

Bringing Similarity Into Light

Experiencing Similarity and Dilations Using Shadows

Matthew Engle



Grades
7 - 8

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Lesson Overview

In this lesson students use shadows to explore similarity and dilations. They are drawn in by a deceiving shadow on the wall when they walk in, and we discuss. Then they explore ratios of measurements found when casting the shadow of a triangle onto the ground, realizing equal ratios. Students are naturally drawn to shadows and love diving in and playing around with them! This also appeals to each learning modality. They will take away conceptual understandings that will leave them regularly wondering deeply about the world around them. The beauty of this framework lies both in its simplicity and the plethora of other topics that tie into it later. Because this structure can be revisited in so many other contexts, students see the beautiful interconnectedness of mathematics. Finally, students will have the opportunity to use shadows as a metaphor to understand differing perspectives so we can make the world we live in a more understanding, cooperative, loving world.

Prerequisite Knowledge
Prior to the lesson, students should be able to:

- Understand the concepts of equivalent ratios and proportionality, especially that the decimal representations are exactly equal.
- Understand the notation for parallel lines and congruent parts of a figure
- Use string and a meter stick to measure length to the nearest tenth of a centimeter
- Use a calculator to find various ratios of measurements they take

Recommended Use
This lesson is best used as an introduction to similar figures, before students have formally learned about them or scale factors. It can be extended in the next lesson to the formalities of scale factors, the one-point dilation method, and setting up and solving proportions based on the ratios in two similar triangles. Future topics where this framework can be revisited are listed in the Extensions section. I don't usually introduce the formal vocabulary until the next day.

CCSS Content Standards
Overall: CCSS.MATH.CONTENT.7.G.A, CCSS.MATH.CONTENT.8.G.A
Specific: CCSS.MATH.CONTENT.7.G.A.1, CCSS.MATH.CONTENT.8.G.A.4

CCSS Practice Standards
MP2, MP3, and MP6

Student Outcomes
Students will be able to understand which parts of separate or nested similar triangles are proportional, especially those necessary for the one-point dilation method. They will recognize that the corresponding sides being parallel makes the ratios hold, and that we can find different distances or angles in the dilation to still preserve those ratios. Lastly, students will recognize the difference between a *dilation* and a *shadow*.

Essential Questions

Students will be able answer questions like these:

- How do we cast the shadow of an object, like a triangle, to keep its shape the same?
- If we hold the triangle horizontal and move it closer to the floor, how must the light be adjusted to keep the shadow the same as it was before?
- What happens if the triangle is *not* held parallel to the ground?
- If you have a triangle and its dilation, and you know some corresponding side lengths, what are the other side lengths?

Lesson Setup

Time Required

Preparation Time: 1 hour, plus time to gather materials

Class Time: One 45-60 minute period

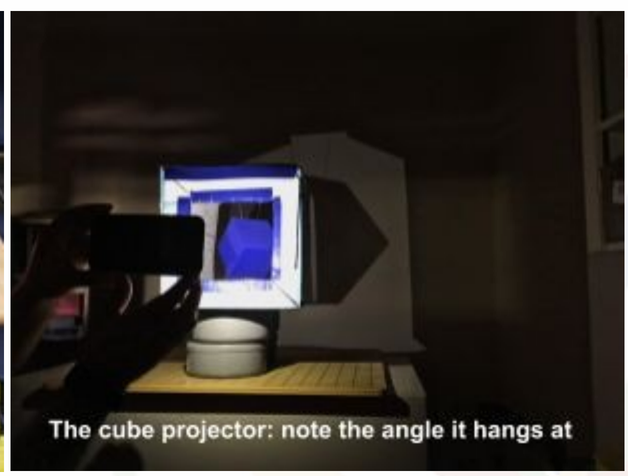
Materials

Masking or duct tape, flashlights (with a wide beam), string (one 5-6 ft length per group), meter sticks (one per group), activity handouts, calculators

If you want to make a box "projector" for the cube in the Intro Activity: cardboard box, fishing line, hot glue, plastic cube from a geometry set, pin or scissors to poke holes, and a flashlight. The key is to hang the cube so that one edge (not side) of the cube is parallel to the ground and turned clockwise (from the top) at 45°. Instead of someone holding the light, as in the photo, I set it up on a table with a light shining from directly behind and draped with a cloth so it's true form remains a secret..

Alternatively, someone can hold a flashlight and the cube for you, with the cube covered to keep it hidden.

Materials



The cube projector: note the angle it hangs at

Classroom Setup

Initial Setup **(20 minutes)**: Move desks to create a large empty floor space in the center of the room. The most important part of this lesson is for the classroom to be dark, as it really creates the aesthetic to instill a sense of wonder and excitement into students. So do whatever you need to do in order to cover windows and make the room safe to walk around in. And of course, make sure to do this within accordance of school policies. You'll also need to pre-select groups of 3 for Explorations 1-3. If your total doesn't divide by 3, make 1 or 2 groups of 4.

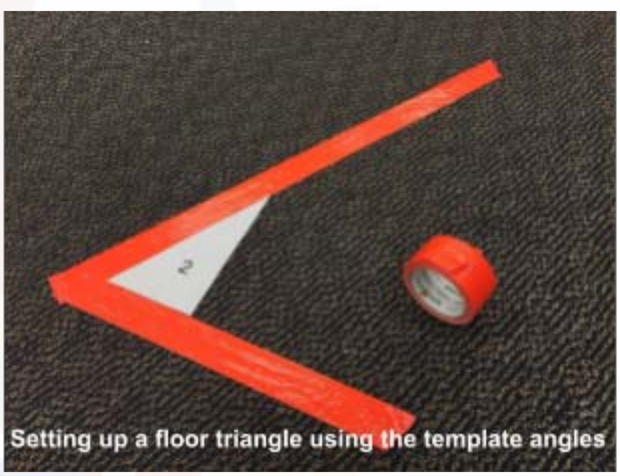
Intro Activity **(10 minutes)**: Find a space to project the shadow for the intro lesson. The best option is an open wall students can see when they walk into the classroom. You can either quickly build a basic "projector" like the description & photo on page 3, or have someone in a hidden place holding the cube steady with the flashlight.

Exploration 1 **(30 minutes)**: Print and cut out the triangle templates included in the handouts. Trace and cut the small triangles out of cardboard, and cut out another copy of Triangle 1 for each group for Exploration 3. Outlines of their larger counterparts will be taped around the floor of the room, spread evenly around. You will have to make the tape triangles larger than the printouts, so just use the angles to create larger versions. I recommend the largest side be 2.5-3 feet long. I use various colors of duct tape to distinguish them for reference, but you can also just write numbers on the tape. If you have more students than can fit in 6 groups, you can just create doubles of some of the cardboard & tape triangles, maybe at different sizes.

Exploration 2: Print copies of the Exploration 2 tables, one for each group.

Exploration 3: One cardboard copy per group of Triangle 1 (the equilateral one)

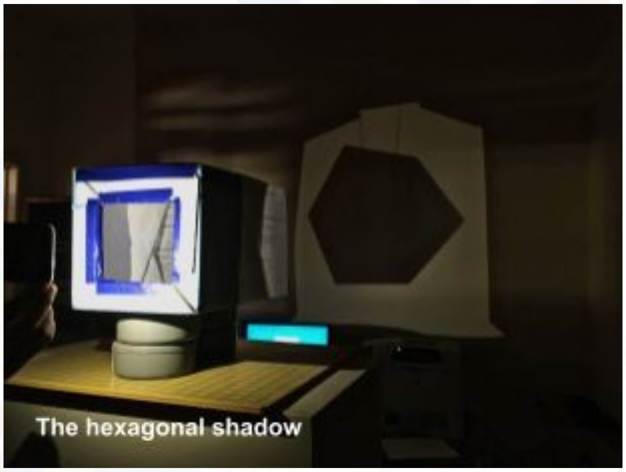
Exit Ticket: Print one copy per student



Setting up a floor triangle using the template angles


The learning space for this lesson

Intro Activity: Deceiving Shadows [5 minutes]



Timeline
Settle In - 2 minutes
Discussion - 3 minutes

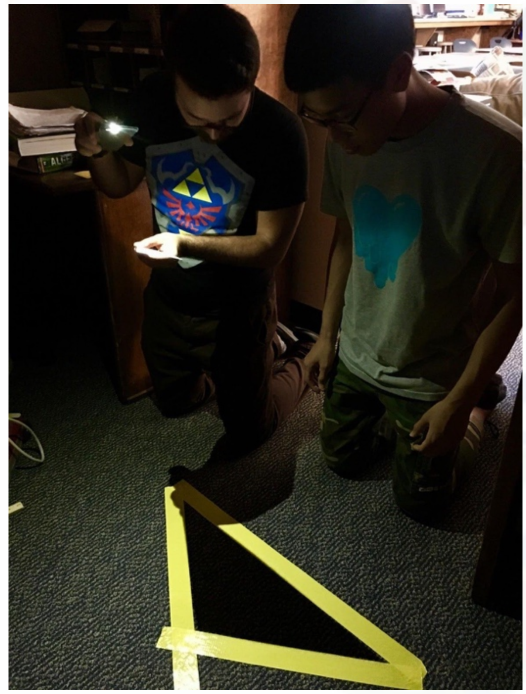
Description
Students walk into a dark classroom, with only a hexagonal shadow displayed on the wall. This comes from a cube, but is hidden so they don't know. Once students settle down you can have them sit on the floor while you ask what figure is casting this shadow. Take responses, likely most will just say hexagon. Have them close their eyes, then shift the cube so that it shows just the square shadow. Then have them look and guess again with this new information. You are likely to get a correct guess this time, if you didn't before. You can now just move the cube around freely and they will see that it's a cube.

While still casting the cube's shadow and moving it around, ask students:
- What made it more clear on the second shadow?
- What made it obvious once I started moving it all around?

Keep an eye out for responses about the position of the cube, the angle of the light, or distances. If that doesn't come up, it's okay because we will get more at that in Exploration 1.

Say something like "Perception is deceiving. Only once you saw multiple shadows were you able to put the truth together."

Exploration 1: That's My Triangle! [10 minutes]



Timeline
Split & Find Triangle - 3 minutes
Discussion - 7 minutes

Description
Split students into their pre-selected groups. Once in their groups give each group a cardboard triangle and a flashlight. Explain to them that their job is to find the triangle on the floor that best matches the shadow cast by their given triangle. You can model this for them, as shown at the right.

Send them on their way finding their triangle, and let them play around for a minute once they get there.

Next, have a discussion with the class. The essential questions to ask here are:

- Can you find more than one way to make your shadow fit?
- If so, what matters for how you hold the flashlight or the triangle?

Post these on the whiteboard while they are playing (shine a light on it to see, they love that!) or display them anyway that works in your room. Give individual think time first, and then have them play & discuss. Finally, come together as a class and record their responses.

We want to get at helping students realize what matters in our projection to keep the triangles similar. They are developing a sense for invariance. Look out for the following conclusions:

- It is possible to fit the shadow more than one way.
- The positions of the light and the triangle matter together. That is:
  - If you move the light up you must move the triangle up as well, and vice-versa.
  - If you shine the angle at a different angle the triangle must shift over to match.
- The easiest way to fit is if the triangle is parallel to the ground
  - Students will play with the angle of the triangle, but for now we focus on parallel

If any of these aren't brought up, you can form them into questions and ask students.

Exploration 2: *Why* is That My Triangle? [25-35 minutes]

Timeline

Role Descriptions - 2 minutes
Measuring & Recording - 15-25 minutes
Discussion & Example - 8 minutes

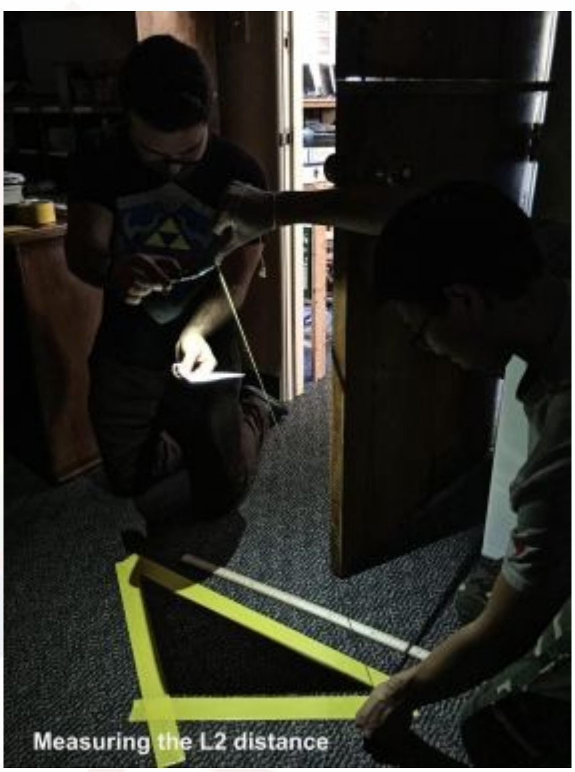
Description: Part 1

Tell the students we will now be exploring more precisely what makes their triangles cast matching shadows. Then hand out the worksheet and have students read it carefully to make sure they understand what to do.

Model for them how to do the measuring and which distances are what, while a helper holds a light and a triangle. It is really important to model the measuring process carefully with them and to strongly emphasize that precision is *really* important! It saves a lot of headache later.

Describe the roles below, and let them decide who will do each.

- **Light & Triangle Holder:** Hold the light and triangle *very* steady, and once you start measuring don't move!
    - For groups with four people, this role can be split in two.
- **Measurer:** Use a piece of string at the length the required distances, and then carefully move it to the meter stick to measure the lengths.
- **Recorder:** Record the measurements coming from the measurer in the tables.


Measuring the L2 distance


Transferring the string to the meter stick

Notes

This is a different look at similarity, and a more modern one. It essentially focuses on learning similarity via transformations, which is a much more intuitive first experience than the usual.

Make sure to go around the room, helping groups hold things steady, measuring correctly & precisely, and understanding how to fill out the worksheet. Some groups will finish earlier than others, so ask them if they notice anything interesting. The ratios in the right columns should all be suspiciously close. If any groups are taking a while, make sure they at least get through the Triangle Sides and Trial 1 at minimum.

If you have a shorter class period, you can just do the Triangle Sides and Trial 1 (15 minutes).
If you have more time do at least the Triangle Sides, Trial 1, and Trial 3 (20-25 minutes).
I only get to Trial 2 sometimes, with speedier groups. But seeing all of these ratios line up is really cool!

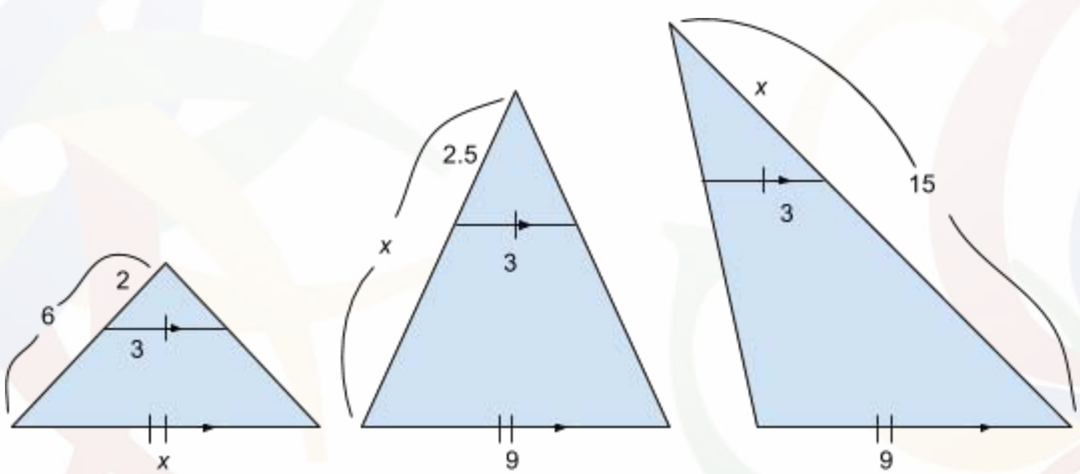
Description: Part 2

Once students are done with filling out their tables, pull them together as a class. Ask:

- What did you notices from the ratios in your tables?
  - Mostly they will notice that they are all (or should be) near a single value
- Were we able to measure *perfectly*? What contributed to error in our measurement?
  - There is error, it was hard to hold things perfectly, we got tired, etc.
- If there *wasn't* any measuring error, what would you expect?
  - Someone should mention that maybe the ratios would be exactly the same

Write on the board: $\frac{L2}{L1} = \frac{T2}{T1}$. Tell students that this is *always* true in this situation!

Draw these three diagrams *without* numbers first and make sure students connect that this is a 2D depiction of what they just did. Then I put in the measurements for the first diagram and we solve it together. I repeat this for the second and the third. Note that the 3 and 9 must stay the same because it is the same triangle side casting the same shadow, just in a different way.



Exploration 3: Hey! That's *Not* My Triangle! [5 minutes]

Timeline
Force the Shadow - 2 minutes
Discussion - 3 minutes

Description
To wrap up the lesson give all groups an equilateral triangle, and ask them:

- "Can you *force* the shadow to fit? I claim that it can be done!"

Help individual groups if they need it, and once they have made it work come together again as a class. Discuss *dilation* vs. *shadow*: a **dilation** is an exact copy but just resized, while a **shadow** is merely a projection that may not look the same.

Discussion: Math Life Lessons
Math is full of life lessons! With the cube or the equilateral triangle, only when we had enough shadows did we get a clear understanding. Think of an issue you care about.

- In what ways is our perception or viewpoint on an issue like a "shadow"?
- How are other people's viewpoints like other "shadows"?
- So understanding the truth is best done by seeing more "shadows". Perception is not always reality! Everyone's perceptions *together* are closer to the truth than our individual perception alone. We need to strive to *understand* each other's viewpoints so we can grow together in our communities and the world.


Making the shadow fit if the triangles aren't similar

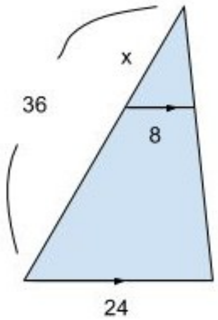
Exit Ticket [5 minutes]

EXIT TICKET

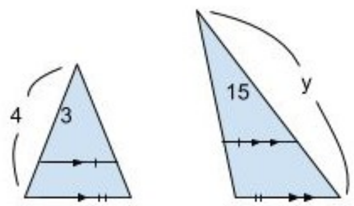
Name ______________________

1. Eric has measured the lengths shown for his shadow. Then he writes into his table the length marked *x* without measuring.

   How does he know what the length should be? What did he write?



2. Kalid and Ramona are in the same group. Kalid casts the shadow and takes the measurements shown in the left diagram. Ramona casts their triangle's shadow differently and gets the measurements shown in the right diagram. What is the value of *y* in Ramona's diagram?



3. Sometime today, go find someone to discuss an important issue you care about (sports, ways to improve the school, whatever!). Share with each other your viewpoints and be ready to share tomorrow one way in which your "shadow" of understanding became more true to the real situation through this conversation.

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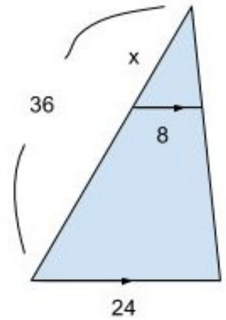
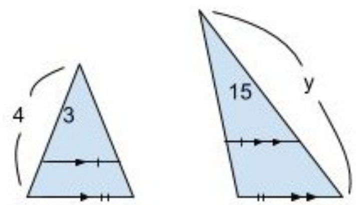




Extensions

Below are some important topics you can revisit with this framework or use some of the ideas we developed from it. This will provide a lot of insight and context. I've used for most of them myself, and this framework makes these ideas so beautiful and understandable. Enjoy!

Next Day:

- Have students share or write a journal about #3 on the Exit Ticket
- The Exit Ticket could alternatively be given as a warm-up today
- Introduce Vocab: similar figures, scale factor, nested triangles, and one-point dilation
- Give follow-up problems using various diagrams, such as the diagram in the Exploration 2 Worksheet (one-point dilation) and the 2D versions (nested triangles) in the examples

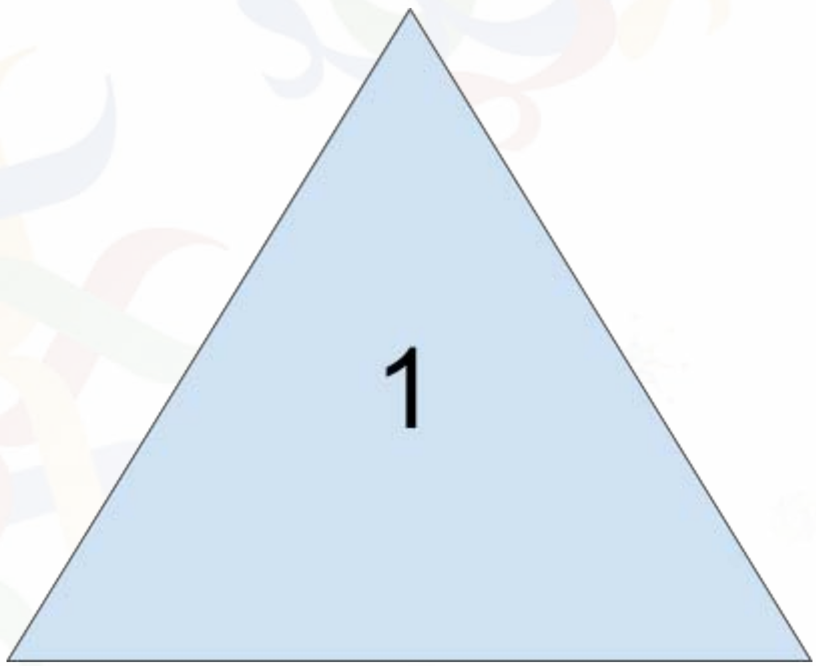
Near Future:

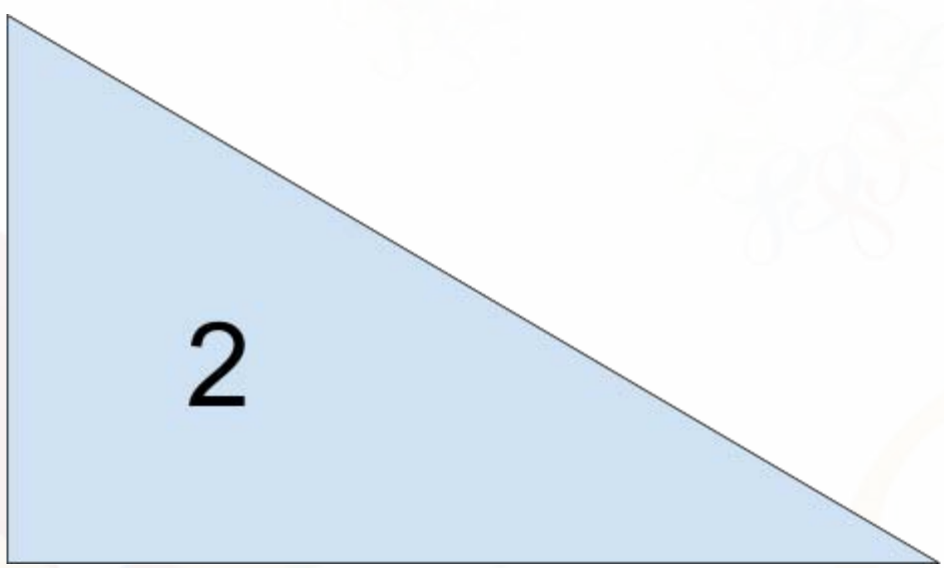
- Practice one-point dilations with other polygons
- Give a project on one- and two-point perspective art
- Extend scale factors to area and volume
- Extend into more complex nested triangle problems
- Points to infinity - Project a dilation and get further and further away; the further you get the closer the figure gets to *congruent*! What if you go infinitely far away?

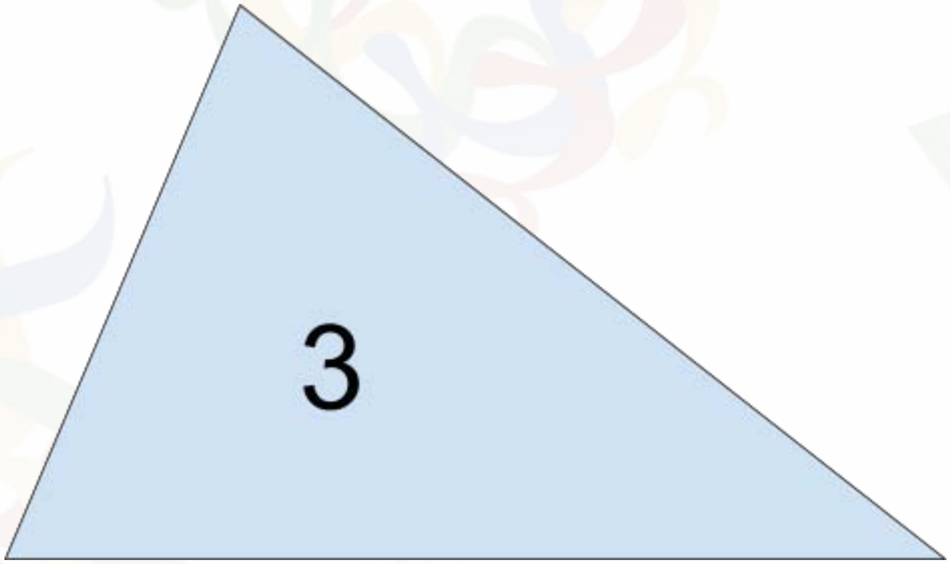
A Bit Further: High School Geometry Topics

- Parallel lines and transversals
- Triangle midsegment theorem
- Conic sections (similar to Exploration 3, but with circles!)
- Understanding dimensions: shadows move down a dimension, studying hypercubes, read the story *Flatland* by Edwin Abbott

Student Triangle Templates



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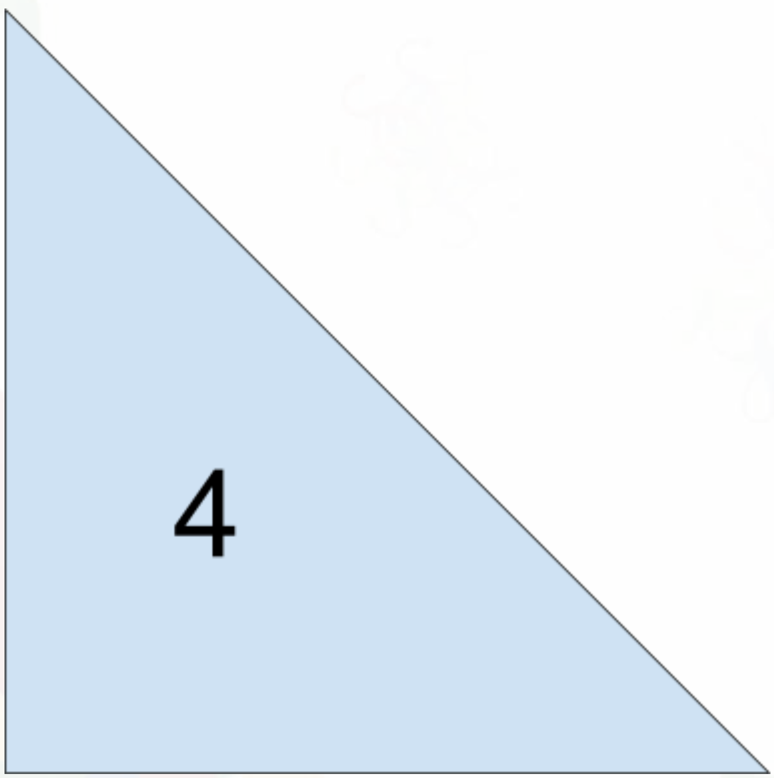
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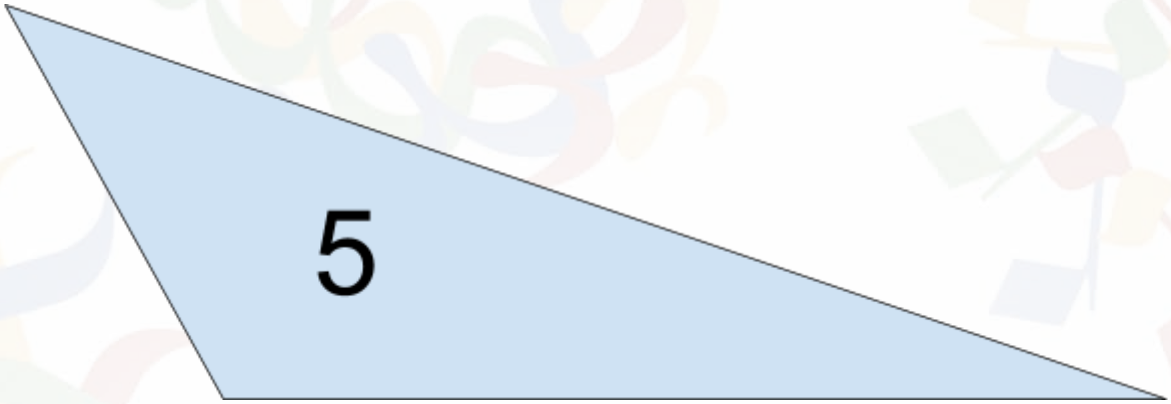
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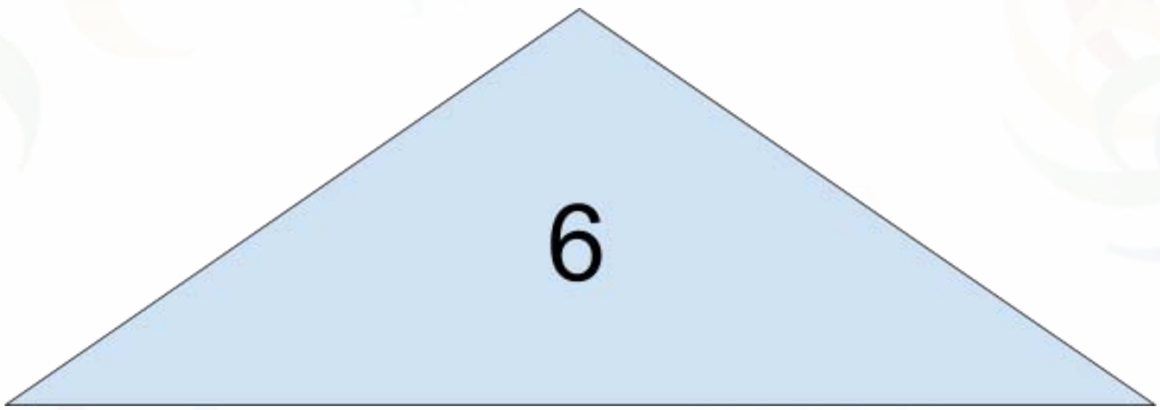
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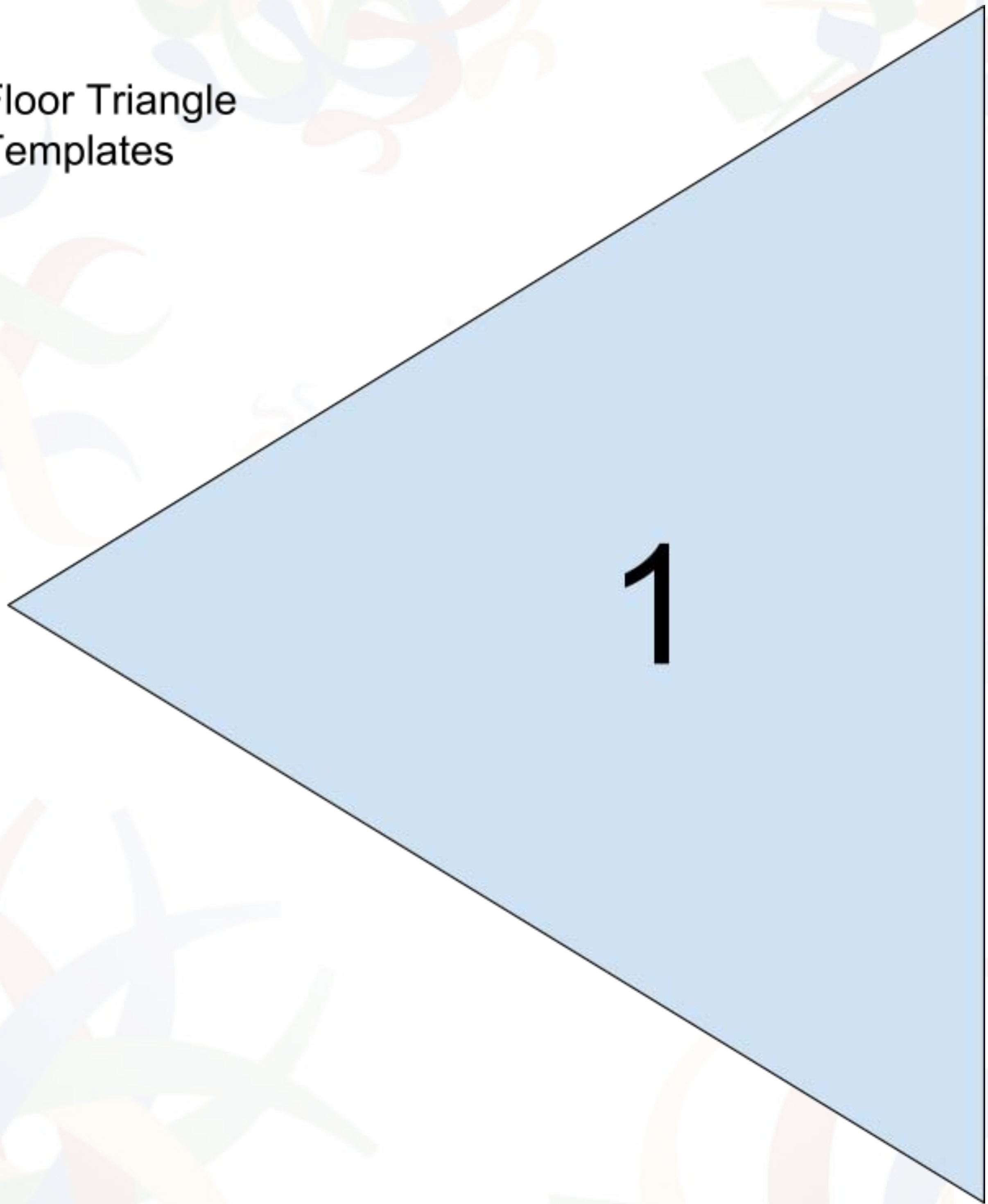
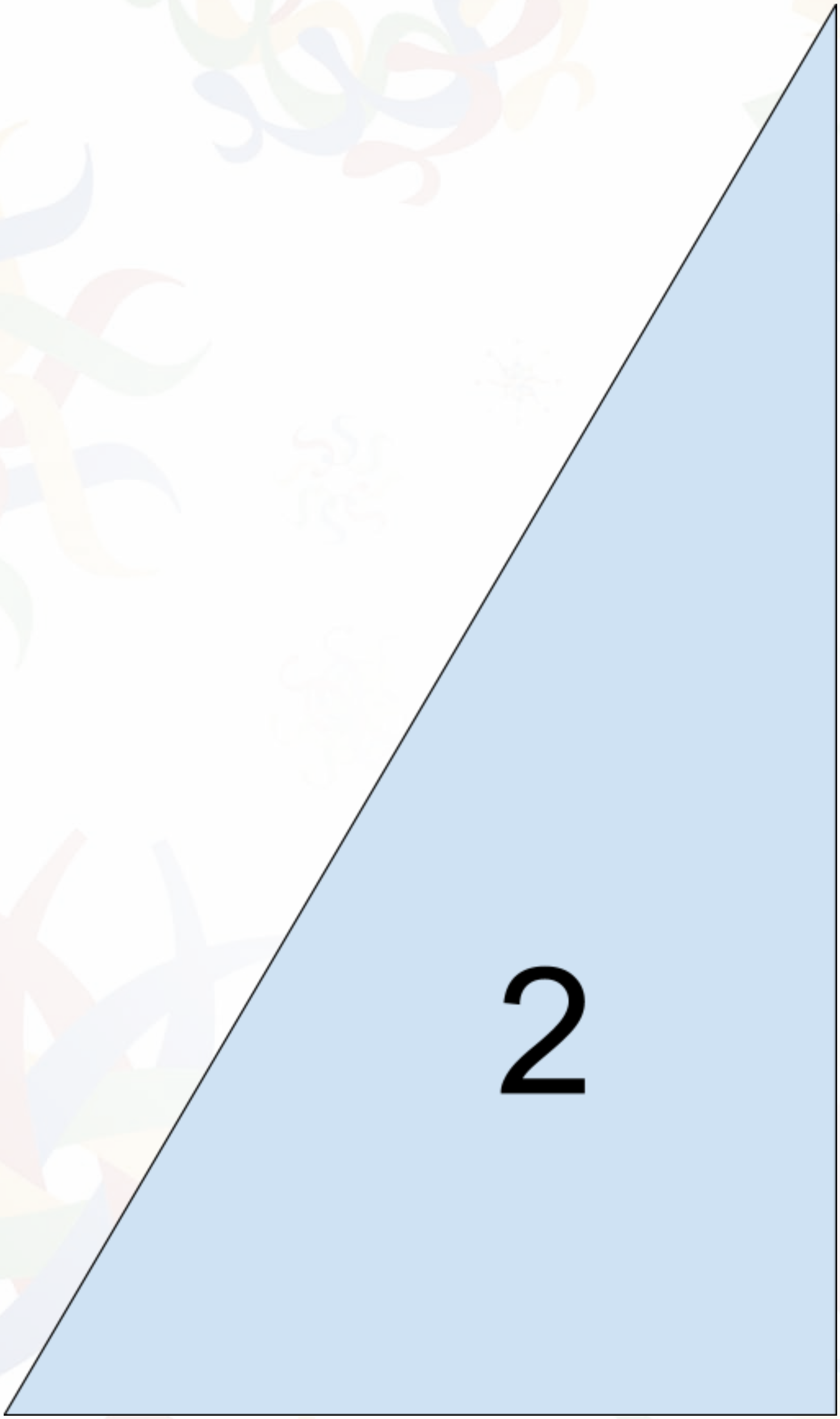
Floor Triangle
Templates

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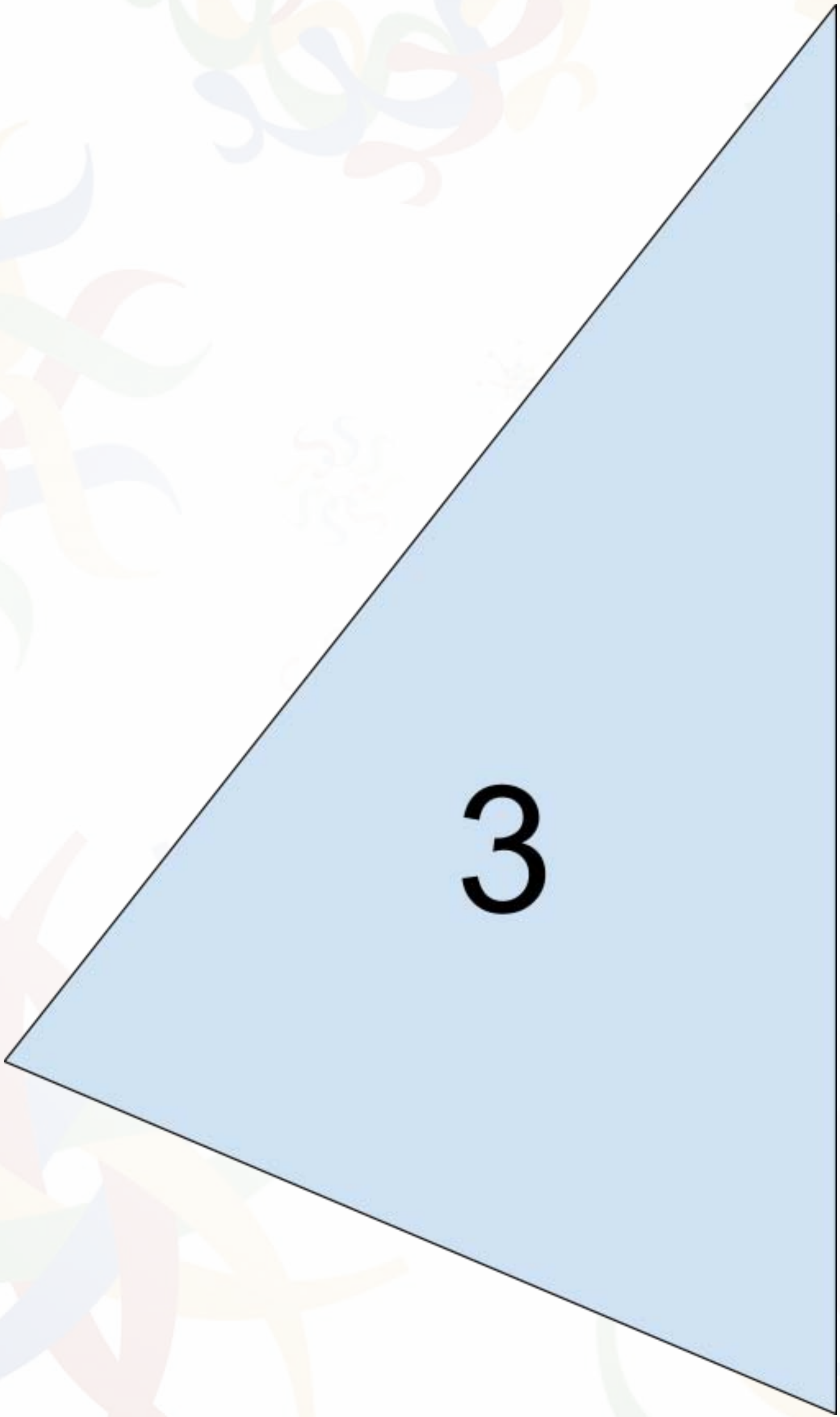
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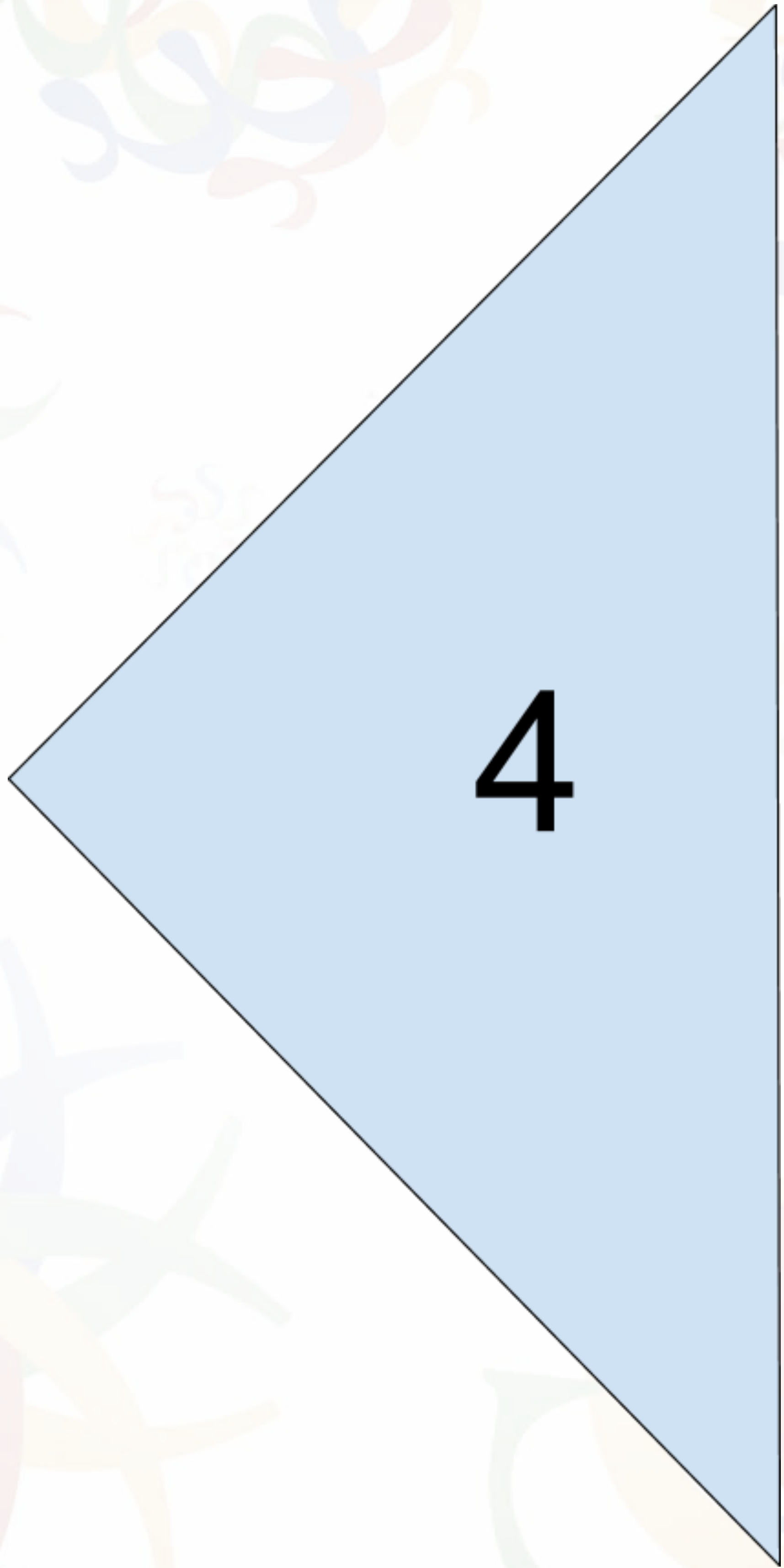
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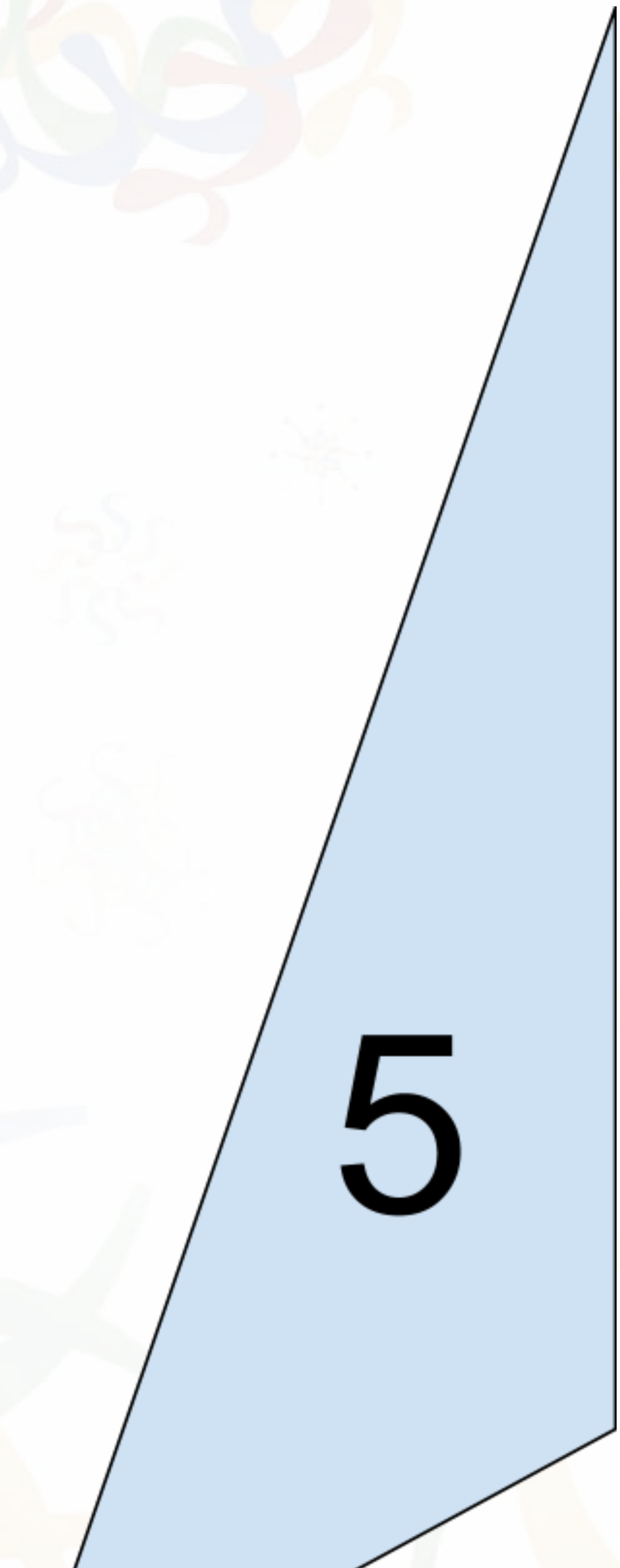
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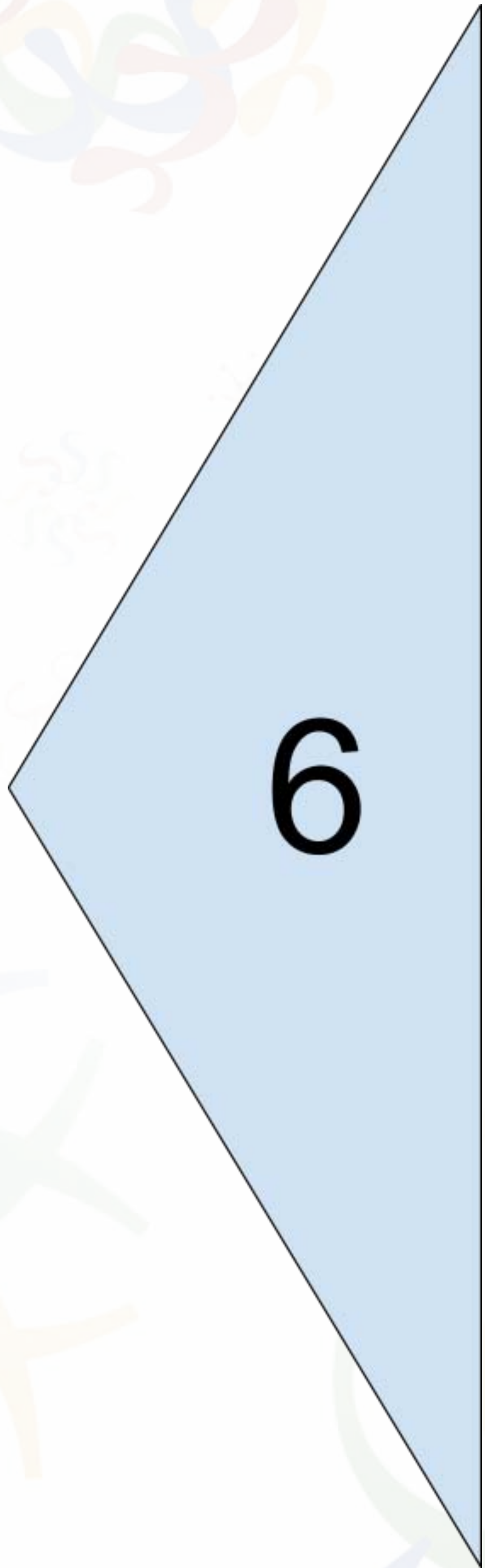
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Why is That My Triangle?

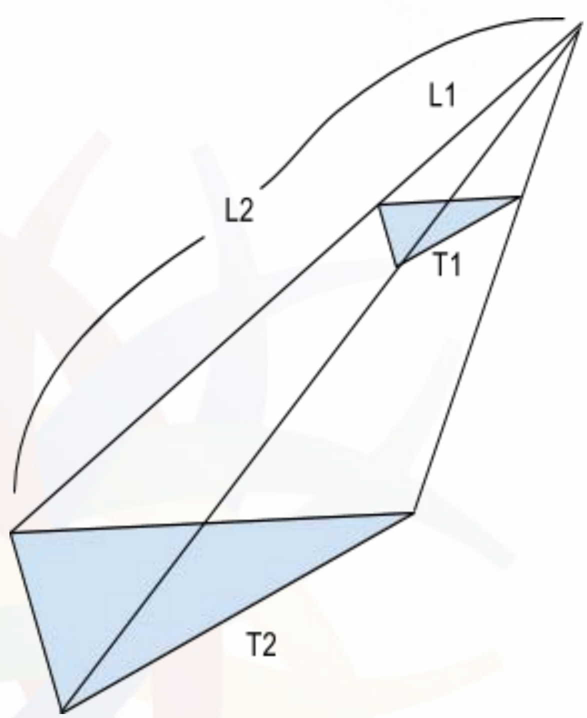
1. Fill out the following tables based upon the measurements defined below.
2. Measure the triangle sides first, followed by Trials 1, 2, and 3 for beam lengths.
3. Write measurements in decimal form, rounded to the nearest tenth of a centimeter.
4. Round the ratios in the right columns to the nearest hundredth.

T1 - Length of a side of your cardboard triangle
T2 - Length of the matching side of its shadow in the tape

L1 - Distance from light to your triangle vertex
L2 - Distance from light to matching shadow vertex

Triangle Sides

	T1	T2	$\frac{T2}{T1}$
Side 1			
Side 2			
Side 3			



Trial 1 - Hold the light directly above with the triangle parallel to the ground

	L1	L2	$\frac{L2}{L1}$
Vertex 1			
Vertex 2			
Vertex 3			

Trial 2 - Do the same, but find a different distance that works!

	L1	L2	$\frac{L2}{L1}$
Vertex 1			
Vertex 2			
Vertex 3			

Trial 3 - Shine the light from a different angle, with the triangle still parallel

	L1	L2	$\frac{L2}{L1}$
Vertex 1			
Vertex 2			
Vertex 3			