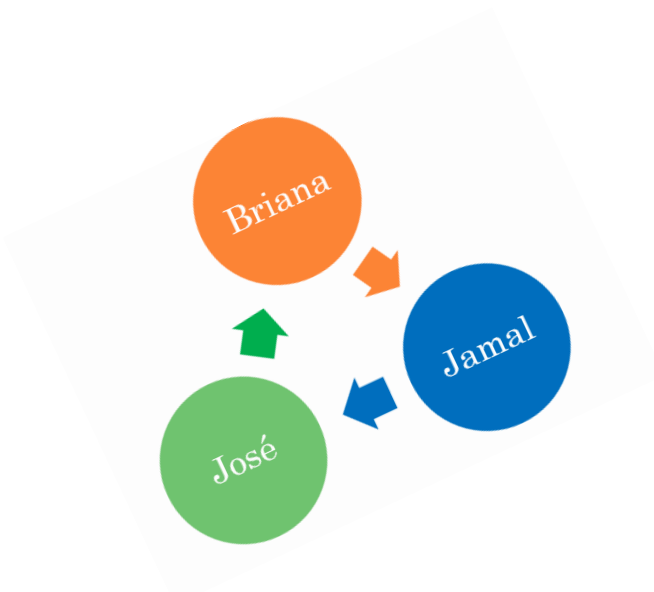
NATIONAL MUSEUM OF MATHEMATICS

THE 2016 ROSENTHAL PRIZE

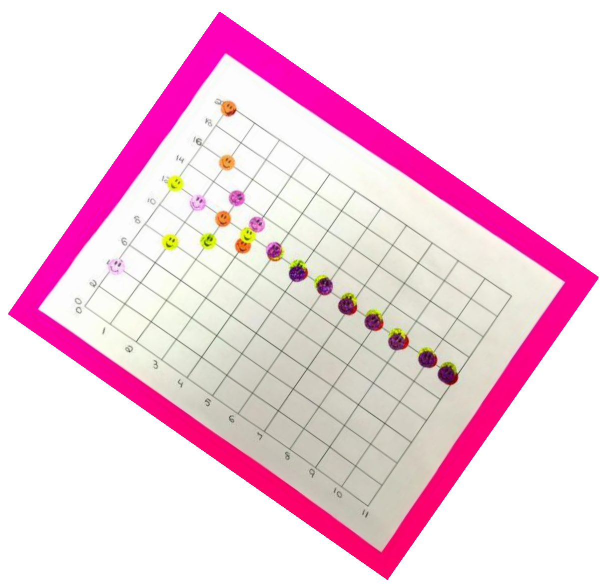
for Innovation in Math Teaching

Pass the Candy: A Recursion Activity

Maria Hernandez



Iteration	Jamal	Briana	José
0	20	10	4
1	12	15	7
2	9	14	11
3	10	11	13
4	11	11	12
5	12	11	11
6	11	12	11
7	11	11	12
8	12	11	11
9	11	12	11
10	11	11	12
11	12	11	11



NATIONAL MUSEUM OF MATHEMATICS

Table of Contents

Lesson Goals ... 3
Student Outcomes ... 3
Common Core State Standards ... 3
Prerequisite Knowledge ... 4
Time Required ... 4
Materials ... 4
Pass the Candy Part I: The Iteration Phase ... 5
- Introduction to the Activity ... 5
- Student Conjectures and Questions ... 5
- What to Do with an Odd Number ... 6
- Iterate: Continue the Process ... 6
- Recording Results ... 6
- An Important Question about Equilibrium ... 7
- Common Student Misconceptions ... 7

Pass the Candy Part II: Graphing the Data ... 8
- Each Group Creates a Graph ... 8
- Share Graphs, Questions and Observations ... 9
- Common Student Difficulty ... 9

Pass the Candy Part III: Extensions, Connections and Reflections ... 9
- Extension Questions ... 9
- Extension Resources ... 10
- Connecting to the Future Study of Recursion and Real-World Problems ... 10
- Reflection Questions ... 11

Pass the Candy Tally Sheet ... 12
Pass the Candy Graphing Sheet ... 13

MOMATH
NATIONAL MUSEUM OF MATHEMATICS

Pass the Candy: A Recursion Activity

Lesson Goals

This activity is designed to be used as an introduction to recursion. The big ideas that emerge from the lesson are:

- An example of what can happen in a system when we repeat some process over and over again according to a specific rule.
- An example of what it means for a system to reach equilibrium.

These ideas can be connected to more specific topics related to recursion (see The Common Core State Standards below).

Student Outcomes

After engaging in this lesson, students will:

- Be familiar with the concepts of iteration, recursion and equilibrium;
- Be able to collect data using recursion and a specific rule;
- Be able to represent data graphically;
- Be able to identify asymptotic or leveling-off behavior of a graph.

The Common Core Mathematical Practice Standards	
Model with Mathematics, Reason Abstractly and Quantitatively, Attend to Precision	
The Common Core State Standards for Mathematics	
CCSS.MATH.CONTENT.8.F.B.5 Describe the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing).	By introducing students to recursion, the lesson can be connected to future CCSS. CCSS.MATH.CONTENT.HSF.BF.A.1.A Determine an explicit expression, a recursive process, or steps for calculation from a context. CCSS.MATH.CONTENT.HSF.IF.A.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers

MOMATH
NATIONAL MUSEUM OF MATHEMATICS

Prerequisite Knowledge

Students should be able to divide integers in half. Part I of this lesson can be adapted for students in Grades 3 – 8 with the goal of introducing students to a recursive process and asking them to pose their own questions and make predictions about the outcome of the process. In Part II of the lesson, students should be able to create graphs of bivariate data on a set of axes, including choosing an appropriate scale for the axes.

Time Required

Preparation Time: 1.5 hours   Class time: 50 – 65 minutes Reflection: 10 minutes

Part I Materials

- Pass the Candy Tally Sheet (1 per group)
- Pencil/pen to record data (1 per group)
- Small bags with pieces of individually wrapped bite-size candy (Life Savers or Starbursts) mixed with some small plastic manipulatives, one bag for each student. Each bag should contain an even number of candies/plastic with numbers of items varying from 2 to 20. You may want to use only manipulatives or paperclips instead of candy.

Part II Materials

- Pass the Candy Graphing Sheet (1 per group)
- Ruler (optional) (1 per group)
- Colored poster paper 11” X 14” (optional) (1 per group)
- Small stickers for each student (optional)
  10 – 12 stickers per student. They should be distinct from their group member’s stickers (see Figure 1).

Figure 1: Materials

Teacher Support Materials (optional)

- Pass the Candy Overview Slides – These slides can be used to explain the activity and help guide the flow of the lesson.
- Spreadsheet with recursion scenarios and graphs described in Part III

Pass the Candy Part I: The Iteration Phase [30 -35 minutes]

Introduction to the Activity [8 minutes]

Divide the class into groups. If possible, each group should have three students. If the total number of students is not evenly divisible by three, you may need to form one or two groups of four students. Students should arrange themselves in their groups so that the desks are in a circle.

Each group of students should be given a Tally Sheet and each student should be given a bag of candy. Ask each group to choose a person in the group to be a record keeper. Ask that student to record the name of each student in the group and the number of candies in each student's bag (across the top of the Tally Sheet).

The teacher will decide when an "iteration" occurs. At the instruction "Pass the Candy," students will divide their candy in half and pass one half of the candy to the person on their left, keeping the remaining half for themselves. Have each student count the number of candies in the resulting "new" pile (what he/she kept plus the items passed from the neighboring student) and ask the record keeper to record that number for each group member on the group's Tally Sheet.

Student Conjectures and Questions [5 – 7 minutes]

After the students have performed this halving process one time, explain to the students that they will iterate (repeat this process over and over again). Then ask them:

- What do you wonder about the process?

Wait for the students to suggest some questions and then perhaps pose the following questions:

- What do you think will happen in the long run?
- Who will end up with the most number of candies?
- Who will end up with the least number of candies?

You may want to have students write their questions on the board. They should also record their conjectures about what is going to happen as they iterate (repeat the process over and over again).

What to Do with an Odd Number? [3 minutes]

Some students may have an odd number of candies after just one iteration, and they will wonder how to divide their pile of candy in half. You can ask the students "What are the choices?"

Here are some possible scenarios from which they can choose:

1. Keep the extra piece, so if I have 7 candies, I pass 3 and keep 4.
2. Pass the extra piece, so if I have 7 candies, I pass 4 and keep 3.
3. Give the extra piece to the teacher or eat it. This removes candy from the larger system.
4. Ask the teacher for an extra piece in order to have an even number of candies. This introduces more candy into the system.

So that you don't get side-tracked here, you may want to have the students vote between choices 1 and 2 and then have everyone follow the same rule. We will call the choices in the list the "rounding rules."

Iterate: Continue the Process [7 – 10 minutes]

Each student should continue to iterate, dividing his/her pile in half, passing half to the neighbor and counting and recording each group member's new total number of candies.

After about six to ten rounds, students should start to see that one of the following outcomes occurs:

1. Each person ends up with the same number of candies.
2. Each person doesn't have the same number of candies, but each group member's amount doesn't change as we iterate and the difference between any two group member's numbers is at most one.
3. Each person doesn't have the same number of candies, but the system gets caught up in a "loop" where the extra piece or pieces float(s) around from student to student and the difference between any two group member's numbers is at most one.

We call each of these final states an equilibrium state.

Recording Results [5 - 7 minutes]

After each group reaches equilibrium, ask a representative from each group to write the initial starting numbers and the "final" numbers on the board.

For example:

		Group 1	Group 2	Group 3	Group 4	Group 5	Group 6	Group 7
Initial Values		4, 12, 20	4, 10, 18					
Final Values		12, 12, 12	11, 10, 11					

Ask students to explain their entries in the table and ask them to answer questions (about their own data) posed by their classmates. It can be interesting for students to share the different types of "equilibrium" values that occur. This part of the lesson can lead to an interesting class discussion about the divisibility of the total number of candies for each group. You may want to ask students to give examples of total numbers of candies for a given group size that will yield a particular type of equilibrium outcome. For example, "We see that if we have a group of three people and 36 candies, each person ends up with the same amount. Give another example of a total number of candies for a three-person group that would result in each group member getting the same amount."

An Important Question about Equilibrium

Why don't we have to keep iterating if the system has reached equilibrium and each student has the same number of candies?

We want students to think about how many candies they will pass on the next iteration and how many they will receive from their neighbor. When we ask students to think about this, we hope that they will recognize that the number of candies stays stable when the number of candies received from the right-hand neighbor is the same as the number they pass to the left-hand neighbor. In this case, "What's going into the pile is the same as what's going out of the pile" (see Figure 2).

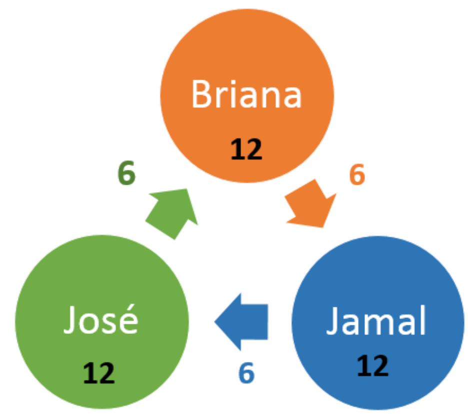


Figure 2: Equilibrium Diagram

Common Student Misconceptions

Some students may have trouble understanding "when to divide their candy in half". For example, suppose you have ten candies and your right-hand neighbor passes six candies to you before you have had a chance to pass the five candies that you were supposed to have passed to your left-hand neighbor. Some students may incorrectly pass eight candies to the left-hand neighbor since eight is half of 16 (the total of 10 "old" candies and six "new" candies). If this happens and students catch the mistake early in the process, they may correct it on their own. If they don't catch the mistake early, they don't need to go back and fix it. They can continue to iterate "correctly" from wherever they are in the

process. This may cause the group to have to iterate a little longer than the other groups because it may take longer for their system to reach equilibrium. To avoid this problem, you can ask students to separate the pile-to-pass from the current pile, and then everyone in the group should synchronize when to pass the pile-to-pass to the left-hand neighbor.

Other possible missteps that can occur:

- Students implement the rounding rule incorrectly. You may want to write the rounding rule on the board with an example such as "Keep the extra piece: If I have 15 candies, I should keep 8 and pass 7."
- Students incorrectly divide their number of candies in half during the process. If students miscount, you may remind the whole class to double check their numbers before they pass the candies. You can also suggest that they make two equal piles of candies in front of them as they are dividing their candies in half so they can see whether they have divided their total pile correctly.

Pass the Candy Part II: Graphing the Data [20 – 30 minutes]

Each Group Creates a Graph [15 – 20 minutes]

The students should graph the number of candies for each group member on the vertical axes and the number of iterations on the horizontal axis.

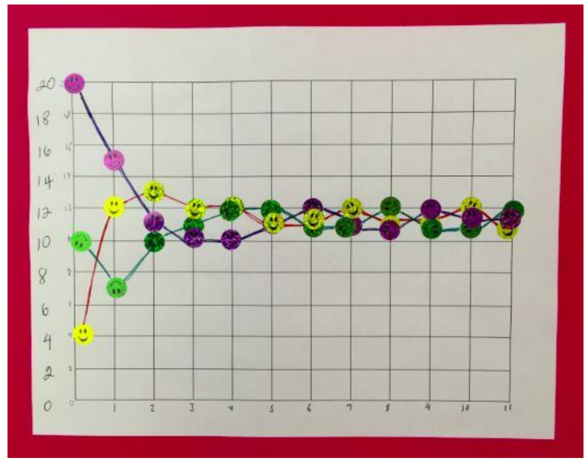


Figure 3: Sample Graphs — Initial Value [4, 12, 20] and [4, 10,20].

Each group of students can create a graph using the *Pass the Candy Graphing Sheet*. These can be displayed around the room. Another option is to glue the Pass the Candy Graphing Sheet onto a colored piece of poster paper. Instead of having students plot the data using different colored pens or pencils, you may want to provide each student in a group with a set of distinct small stickers (10 – 12 each) to use as their data points. Each student will graph his/her data using a distinct sticker. Even though the data should be represented by discrete points, it may be helpful to "connect the dots" to see the behavior of each student's graph (see Figure 3).

Share Graphs, Questions and Observations [5 – 10 minutes]

After giving the entire class a chance to look at all the graphs, ask them to share any observations or questions that they have. These graphs can help students see how each individual member's graph compares to the other members' graphs. Some graphs may have data points that increase and then decrease or vice versa before reaching equilibrium. Students should be able to describe the "end behavior" of the graphs using their own words such as "The graphs level-off." Some students might say that the graphs "bounce back and forth between two values." These graphs can introduce students to the idea of the "long-term" behavior of a graph. These graphical representations can also lead to the important concept of a horizontal asymptote. Another important idea is for students to consider whether the graphs should be distinct points or connected graphs, making a distinction between discrete graphs and continuous graphs.

Common Student Difficulty

Some students have trouble "starting" their graphs, because they don't understand that they can put their first point on the vertical axis, thinking of the initial values being associated with the 0^{th} iteration. You can encourage students to "start" their graphs with an x-coordinate of 1 if they are confused about using 0 as their starting x-coordinate.

Pass the Candy Part III: Extensions, Connections and Reflections

Extension Questions

After students have shared their results, you can ask them to think about other scenarios and pose some of their own questions. Some possible questions that might arise are:

- What would happen if we change the rounding rule? How will this change the outcome?
- How long will it take to reach equilibrium? How do the initial values affect this time to equilibrium?
- Could we predict the long-term outcome if we have different starting values?
- What would happen if each person passes one third of this/her candy instead of half the candy?

Extension Resources

A spreadsheet is provided with the lesson. This spreadsheet is designed to simulate the scenario described in the main lesson, but it can be adapted to explore some of the questions posed in the Extension Questions section above. The rounding rule used in the spreadsheet is the "keep the extra piece" as described in the main lesson. Each tab in the spreadsheet gives an example of either a three- or four-person group. For a three-person group, the various sheets provide an example of the three possible outcomes as described in the main lesson. An example from the spreadsheet is shown in Figure 4 along with the graph of the values for each group member.

Iteration	Jamal	Briana	José
0	20	10	4
1	12	15	7
2	9	14	11
3	10	11	13
4	11	11	12
5	12	11	11
6	11	12	11
7	11	11	12
8	12	11	11
9	11	12	11
10	11	11	12
11	12	11	11

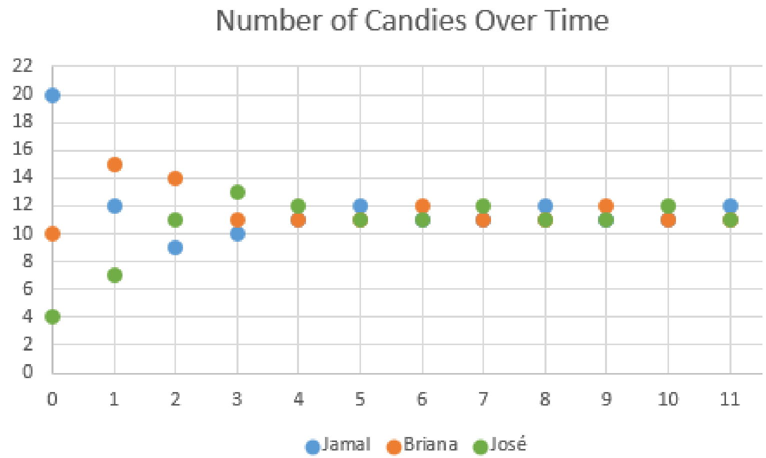


Figure 3: Example: Data for a three-person group where the total number of candies is not evenly divisible by three.

Connecting to the Future Study of Recursion and Real-World Problems

This activity sets the stage for the study of recursive equations and the behavior of the graphs of those equations. For example, students can explore the idea that a sequence that is created by repeated addition can be described both by a recursive function and an explicit function. More specifically, the sequence {2, 7, 12, 17, 23 …} can be expressed

using a set of recursive equations: $a(0) = 2$ and $a(n) = a(n-1) + 5$ or as an explicit function, $y = 5x + 2$.

Some examples of real-world problems that can be connected to this activity are:

- How does a repeated dosage of a drug, like a pain reliever, get metabolized in the body? The kidneys eliminate some of the drug throughout a time period (say, every four hours) and then the patient ingests another dose of the drug. The repeated dose is prescribed in hopes that the amount of the drug in the system reaches an equilibrium value or a therapeutic level in the body.
- How does pollution move through various connected bodies of water? Since water flows from one body of water to another, the pollution moves along with the water. Students can explore what happens to the amount of pollution in each body of water if they know the flow rate of the water from one body to the next and how much of the pollutant is being added to one or more of the bodies of water.
- How does a disease move through a population? People are considered to be in one of the following groups: susceptible, infected, or recovered. As the people contract the disease and then recover, they move from the susceptible group to the infected group, and then to the recovered group.

- How does an ingested substance like lead move through your body? If a person ingests lead, the lead moves from the blood to the organs to the bones.

Reflection Questions

After students have engaged in this activity, you may want to pose some reflection questions. You can ask students to address these questions outside of class as a homework assignment and then share their reflections with each other in class the following day. Here are some possible questions:

- What did you find interesting about the activity?
- How do you think the mathematical ideas in this activity connect to other math topics you have studied?
- What kinds of questions would you want to explore after having participated in this activity?

Pass the Candy Tally Sheet

	Student 1	Student 2	Student 3	Student 4
First Names →				
Start Values # of candies →				
# candies after Round 1 →				
Round 2				
Round 3				
4				
5				
6				
7				
8				
9				
10				
11				

MOMATH

NATIONAL MUSEUM OF MATHEMATICS

Pass the Candy Graphing Sheet