MoMath Workshop: Topological Crochet

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Introduction

This handout is prepared for the MoMath workshop series *Topological Crochet*. The purpose of the series is to share the powerful method, by which we start a project by building a ribbon graph with foundation chains. This initial set up totally determines the topology of our final work. The key steps in these projects are:

- Build a foundation chain ribbon graph by chain *twists* and stitch *joins*, which we will cover later. Use stitch markers to hold the joined stitches together.
- Crochet a round on top of the chain graph to solidify the structure. Each location that the stitch markers hold in place should be crocheted into once. All stitch markers are safely removed at the end of this step.
- Use our personal taste and imagination to design rounds of stitches to achieve geometric and artistic effects.
- Crochet along the boundary of our work around wire to strengthen the sculpture.

The first step is the key to the whole project, since it decides the topology. For passionate participants who are newer to crochet, it is advisable to practice chain stitches at home before the workshop. Many online resources, such as [5], are available for learning crochet from scratch.

A *twist* in this handout always means *180-degree* unless otherwise noted.

Material Needed

There is no restriction as to the type of *yarn* we can use. But it is much easier if we avoid extremely dark colors at the beginning, as it is important for us to see each stitch. For the same reason *at least DK weight is recommended*. Multi-ply worsted weight (non-merino) wool has been working best for me.

In addition to a good *crochet hook* for the hands, some kind of *stitch markers* are essential for topological crochet. They are used to hold the joined stitches together before we can crochet into them. Each place where stitches join requires multiple markers, depending on the valence of that point (we will see examples later).

Blocking pins, though not essential, are very helpful when setting up the foundation chain graph. I make empty temari balls with different sizes and colors to accommodate different projects. *Scrap yarn balls* are good choices too.

Boundary material is useful to give the projects an artistic touch. One can use monofilament fishing line (250lb–500lb weight) with connectors of the matching size. I usually use non-insulated butt connectors. Other types of supportive wire are weed wacker line, or metal wire, for example. The demo projects in this handout were done using half-hard round copper wire, but you can replace it with any material you like.

Möbius Band

Twist and *join* are two basic operations of topological crochet. To practice these, we will make a Möbius band. The idea is similar to Cat Bordhi's knitted Möbius cowl [3] but much less technical in crochet. Step-by-step instructions for this project are shown in Figure 1. We need one *foundation chain* of about 27 stitches (Figure 1(a)). The exact number doesn't matter here. Twisting it and joining the end chain stitches produces a narrow Möbius band (Figure 1(b)). We then add a round of *single crochet* stitches (Figures 1(c)-1(g)) to make it prettier and more stable. Figures 1(h)-1(k) show how to fasten off neatly. The final work shown in Figure 1(l) produces a satisfying result. Figure 1(f) is the key step, where a lot of people make mistake. Make sure you do encounter the blue marker.

(a) *Make a foundation chain of about 27 stitches.*

(b) *Twist it and join the end chains together by a blue stitch marker.*

(c) *Pull the working yarn loop to the front and chain one.*

(d) *Start single crocheting and mark the first single stitch.*

(e) *Encounter joined chain stitch with the blue stitch marker.*

(f) *Crochet into the joined chain stitch and remove the marker.*

(g) *Stop crocheting before the marked first single stitch.*

(h) *Cut the yarn.*

(i) *Pull the yarn end through and remove the hook.*

(j) *Thread the yarn in a needle and insert it in the first sc marked by the white marker.*

(k) *Remove the white marker and pull the yarn end through the last stitch.*

(l) *The final work after weaving in the ends.*

Figure 1: *Making a Möbius band.*

It is important to remember that the two end chain stitches that were joined should *always* be treated as if they were a single stitch whenever we crochet into them. This applies to all joined stitches in all projects for this workshop.

Some other tips that might help:

- Be very familiar with what a foundation chain looks like, when untwisted.
- To twist, it's easier to hold the end with the working yarn with one hand, and twist the chain by twisting start-of-the-chain end. This way we don't need to turn our work back and forth unnecessarily many times to insert the crochet hook back. As shown in Figures 1(a)-1(d), the right end of the foundation chain hardly moves position during the twist and join operations.
- One might notice the constant need to turn the work when performing single stitches. This is a characteristic of this style of topological crochet: the *natural flow* of the chains guides our crochet hooks. Care has to be taken when crocheting so that such flow is respected and no unwanted local twists are created. This can be done by always holding the piece so that
  - you have an unobstructed front view of the segment of the row you are working on and
  - the working yarn is in the back of that row.
- Each crocheter has their preference of whether or not to include the back bumps when crochet into the foundation chain. To reduce unnecessary confusion, make a choice and don't change it in the middle of a project.

Trefoil Knot

The main project in this workshop is to make a surface bounded by the trefoil knot, which is the simplest non-trivial knot. One can find many fascinating mathematical features and artistic implementations of it online. The following are the step-by-step instructions of our crochet sculpture. The third step uses treble crochet stitches (tr) as well as *adding* and *combining* of stitches. Those who are newer to crochet are free to skip this step during their first attempt. In [7] a video instruction of this project is available for participants' reference after the workshop.

- **Build the foundation.** Chain 49 stitches plus a few extra that we will remove later. Remove the crochet hook. Starting from the first stitch, mark every 16th chain stitch (Figure 2(a)), four in total (at numbers 1, 17, 33 and 49). If necessary, secure the first chain stitch in place on a temari ball or a yarn ball. Twist once to secure the second marked chain stitch. Twist again to join the first chain stitch with the third marked chain stitch. Without a temari ball this can be done by joining the *first* and *third* marked stitches with a *360-degree twist*. Place *three* stitch markers at this join (Figure 2(b)). Then twist once more to join the *second* marked chain stitch with the *fourth* marked chain stitch. Place *two* stitch markers at this join (Figure 2(c)). We have built a graph with two vertices, or *branch points* as we will call them, with three edges connecting them. Make sure all the edges twist in the *same* direction.
- **Round 1: remove the stitch markers.** Insert crochet hook into the second join (Figure 2(d)), undo all the extra chain stitches (Figure 2(e)), and chain 2 (ch2) to get ready for round 1 (Figure 2(f)). Crochet round 1 with the following guideline: *double crochet (dc)* at either branch point, *half-double crochet (hdc)* at the two neighbors of either branch point and *single crochet (sc)* the rest (Figures 2(g)-2(j)). In total we will go through six sides of the edges, each with a half-double crochet stitches at each end, and 13 single crochet stitches in between. There will also be three double crochet stitches around each

(a) *Chain at least 49 and mark four stitches.*

(b) *Join the two stitches marked blue with a 360-degree twist.*

(c) *With another twist, join the stitches marked green.*

(d) *Insert the crochet hook back into the joined stitch marked green.*

(e) *Remove extra chain stitches.*

(f) *Start the first round with ch 2.*

(g) *Crochet along the first edge with a hdc followed by many sc.*

(h) *End this edge by another hdc, before the blue marker.*

(i) *Remove the blue marker, dc, and crochet along another edge.*

(j) *Continue till all edges are crocheted into on both sides and end round 1 by a slip stitch.*

(k) *After the first outer edge is finished of round 2.*

(l) *After round 2 is finished and fastened off.*

(m) *Start the boundary round with wire.*

(n) *Join the wire ends before finish.*

(o) *Ready to fasten off.*

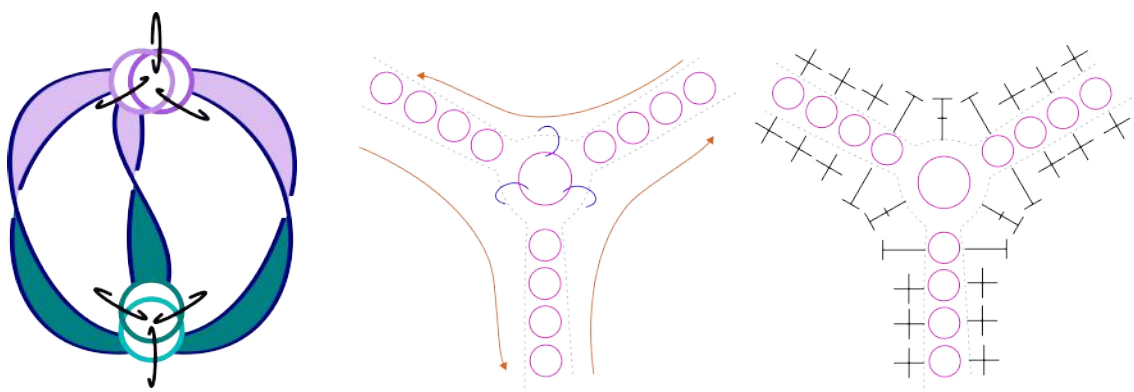
(p) *The final work after fasten off and weaving in the ends.*

Figure 2: *Making a surface bounded by the trefoil knot.*

branch point. Whenever we crochet past a stitch marker, remove that one but not the rest in the same branch point. Use a slip stitch to end the round. For those who are newer to crochet, dc and hdc may be replaced by sc all around.

- **Round 2: beautify.** This step is *optional*. Place the work so that you can trace the trefoil knot. For any of the three twisted edges, there is one side facing outward and one inward. For the outward side we will crochet taller and more stitches and for the inward side the opposite. Double check there are 16 total stitches on each side of the edges. We are going to crochet in a symmetric way. On the outer side, for the eight stitches from the branch point to the center, crochet as follows: *dc for the first four, 2dc for the next two, 2tr for the next two* – I'll call this an *outer half sequence (ohs)*. On the inner side, for the eight stitches from the branch point to the center, crochet: *dc, hdc, sc, sc, sc2tog, sc2tog* – I'll call this an *inner half sequence (ihs)*. Starting from any branch point, we'll crochet the *combined sequence of ohs, reversed ohs (Figure 2*(k)*), ihs and reversed ihs three times* (Figure 2(l)). Fasten off. This round is where the whole project resembles ordinary crochet the most – we choose the stitch arrangement to vary the size or to add other decorative aspects. The participants are encouraged to experiment with different arrangements to achieve different visual effects.

- **Round 3: color and wire the boundary.** This is the most exciting step as our sculpture will gradually reveal its three dimensional look as we work along its boundary. Cut a 30 inch long wire. Here I'm using gauge 20 copper. Use the second color to single crochet around the wire and along the boundary, starting from a point that is close to the beginning of an ohs (Figure 2(m)). This point is chosen so the wire ends will hide better. Stitches can be added when the boundary curvature is high, as long as the choice is symmetric. I choose to *add three on each outer side*. Before finishing the last inner side, join the wire ends and trim the extra (Figure 2(n)). Fasten off carefully (Figure 2(o)). After finishing, adjust the boundary so that it looks smooth and symmetric (Figure 2(p)).



(a) *Stitched marked chain graph.* (b) *Around each branch point.* (c) *Stitch diagram of the round one around each branch point.*

Figure 3: *Illustration of how to setup the chain graph and crochet round one.*

To summarize, in the first two steps we build the foundation chain graph and reinforce it. In the third step (Round 2) we add artistic elements, and in the fourth step (Round 3) we finish the work by decorating it with desired colors and using metal to support the whole surface. The first, second, and fourth steps are essential, while the third step can be omitted completely.

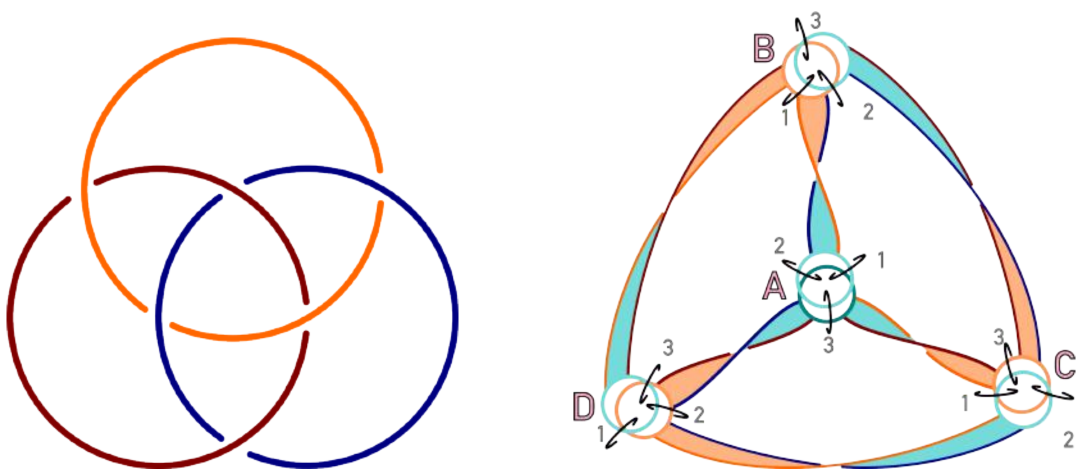
More tips:

- Once again, *extensive stitch marking* is always necessary, even when we think we fully understand. Since there is no natural global coordinate system it is very easy to forget how the edges around each branch point should line up. It is worthwhile to use a stitch marker *between every pair of neighboring edges*.

- Let the wire bend along the boundary without too much intrinsic twist, because when the wire has high twist resistance it'll change the bend and introduce unnecessary tension on our stitches.
- Since the boundary is usually a non-planar curve, when we crochet along it we need to reposition the yarn ball back and forth; a smaller yarn ball is easier to maneuver. Sometimes the stitches are hard to reach; a long Tunisian crochet hook can help. Be extra careful not to miss any stitches.

Borromean Rings

Another simple yet profound example is the Borromean rings. Here we are going to make a nonorientable surface bounded by this link, and it has *chiral tetrahedral symmetry*, which is apparent in each step of the construction. A table of many other polyhedra-based projects can be found in [4]. [2] contains visualization of many polyhedra.



(a) *Link diagram of the Borromean rings.* (b) *Finished foundation chain graph.*

Figure 4: *Diagrams of the foundation chain ribbon graph.*

Figure 8(a) is the link diagram, and Figure 8(f) is the chain graph after applying the two-color algorithm (introduced later), where the front side of the foundation chain is indicated in aqua and back in apricot. We label the vertices (also called *branch points*) with "A, B, C, D," and the stitch markers "1, 2, 3." The stitch markers at the branch point A will be referred to as A-1, A-2, and A-3, and the others are labeled similarly.

1. **Build the foundation.** Let the edge stitch number m. In Figure 5, $m = 9$, but 10 and 11 are both good choices. Prepare two foundation chains with $4m + 1$ and $2m + 1$ stitches, possibly with a few extra. For each foundation chain, starting from the 1st chain mark every mth stitch, five $(1, m+1, 2m+1, 3m+1, 4m+1)$ in total for the longer foundation chain and three $(1, m+1, 2m+1)$ for the shorter chain. With both chains' front sides facing up, join the 1st stitch of the shorter chain with the center marking of the longer chain and put three stitch markers all around (Figure 5(a)). This point will be branch point A. With the help of a cushion ball, make the rest of the tetrahedron by joining the marked stitches appropriately following Figure 8(f). Each edge of the tetrahedron has a 180-degree twist in the same direction. This can be done by making the three side-triangles ABC, ACD and ADB (Figures 5(b)–5(d)). The triangle BCD will be automatically completed. Place three stitch markers

(a) *Secure the first branch point.*

(b) *Make one triangle with all edges twisted.*

(c) *Another triangle.*

(d) *The whole tetrahedron with all edges twisted.*

(e) *At a branch point, chain 2 (for dc), hdc and sc.*

(f) *Finishing one circle.*

(g) *Starting the second circle.*

(h) *Starting the third circle.*

(i) *Finishing the third circle.*

(j) *Place a wire long along a circle.*

(k) *The first circle is done.*

(l) *The final work.*

Figure 5: *Making a surface bounded by Borromean rings with chiral tetrahedral symmetry.*

around each branch point except the ones with the working yarn, where only two are needed. Ideally, the markers are color-coded.

2. **Round 1: remove the stitch markers.** In one of the branch points with the working yarn, undo all the extra chain stitches, insert the crochet hook into the joined stitch, and pull the working yarn loop through the top stitch and chain 2 (ch2). Crochet round 1 with the following guidelines: double crochet (dc) at each branch point, half-double crochet (hdc) at the two neighbors of each dc, and single crochet (sc) the rest. There will be three double crochet stitches around each branch point. Remove a stitch marker whenever we crochet past one. This round has *three* separate circles, each going through *four* sides of the edges.

   Figure 5(e) shows how your work should look after the first few stitches. After four stitch markers are removed and replaced by dc, we are back at the first stitch of this component, fasten off, and cut the yarn (Figure 5(f)). Repeat this for the two other components (Figures 5(g)–5(i)). For those newer to crochet, dc and hdc may be replaced by sc all around.

   The three circle components visit stitch markers *(B-2, C-2, D-2, A-2)*, *(C-1, D-1, B-1, A-1)* and *(D-3, B-3, C-3, A-3)*.

   Like the Borromean rings project, in this one, the final crocheted surface is nonorientable. As a result, one has to crochet sometimes from the front and sometimes from the back side of the foundation chain. The side switching happens when one crosses and removes the stitch markers D-1, D-2, B-1, B-3, C2, and C-3.

3. **Round 2: color and wire the boundary.** Cut three pieces of 22-gauge copper wire with equal length, about 11 inches. Use the second color of yarn to sc around the wire and along one of the boundary circles, starting from any point. Since now the boundary has three simple components, we can close the wire ends before we crochet (Figure 5(j)), making the crocheting part easier. Repeat this for the two other components with different yarn colors. Make sure the wire circle size is the same for all three components. The project is finished (Figure 5(k)–5(l)).

Seifert's Algorithm

This section will briefly review the classic Seifert's algorithm to create a Seifert surface [11]. A *Seifert surface* of a given knot or link is an *orientable* (or two-sided) surface bounded by such knot or link. For example, a disk is a Seifert surface of a circle, but a Möbius band is not since it is famously non-orientable. Seifert surfaces exist for a given knot or link, as Seifert's algorithm below shows. A Seifert surface is not unique, as a circle has at least two Seifert surfaces: a disk and a punctured torus.

The Seifert's algorithm adapted to topological crochet:

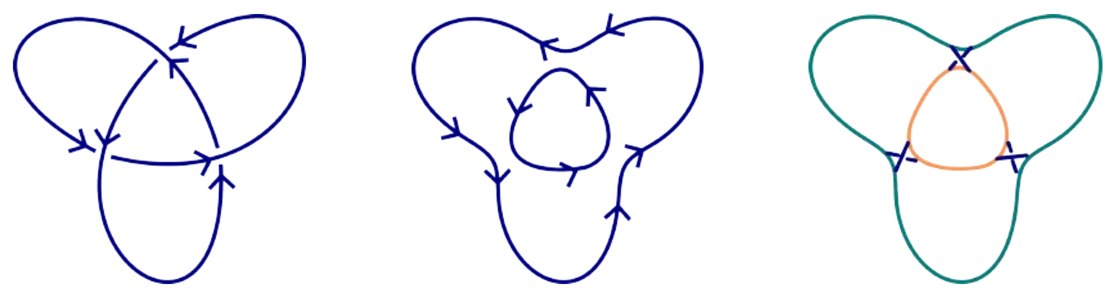
1. Draw the knot or link diagram with arrows.
2. Change each knot crossing in Figure 6(a) into Figure 6(b). The diagram now comprises multiple closed loops that don't cross each other.
3. Collect all the closed loops, each representing a patch in the surface. The patches are either lying on top of or next to each other. Each patch implies a *chain stitch* in the foundation chain graph.
4. For each crossing in the original diagram, attach a twisted band joining the patches - the final surface results from all patches with all bands joining them. In topological crochet terms, this means a *twisted foundation chain* connecting the two corresponding chain stitches.

(a) *Original crossings in a knot or link diagram.*

(b) *Crossing deformed into two parallel strands.*

Figure 6: *Modification of crossings in Seifert's algorithm.*

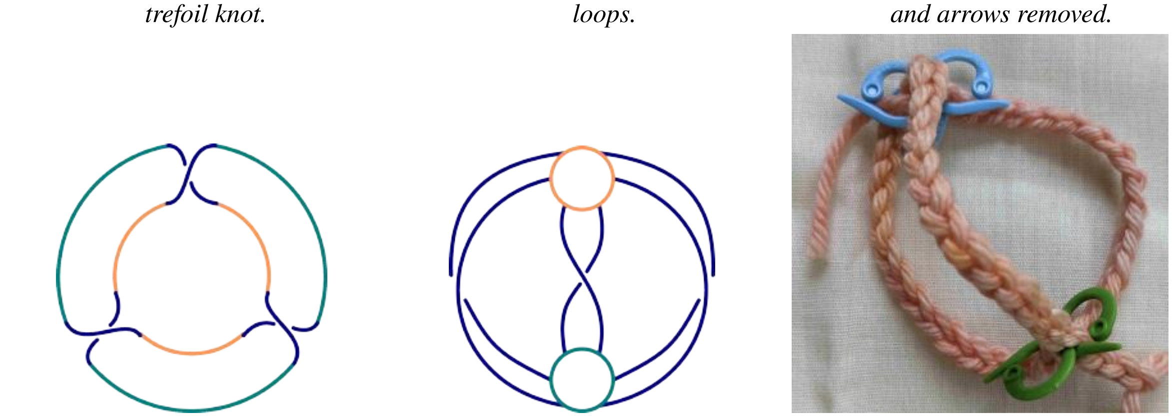
Let's see how it works from an example in Figure 7. We start from a trefoil knot with consistent arrows in Figure 7(a). Modifying all three crossings leads to two closed loops in Figure 7(b): an outer one and an inner one. These two closed loops are two patches in our work to be connected in a certain way. To find how they are connected, we then remove the arrows and put all original crossings back in the diagram as in Figure 7(c). These crossings connect the two patches as in Figure 7(d). Each crossing can be considered as a band connecting the two patches with a 180-degree twist. All three bands twist in the same direction due to symmetry. We can draw the final diagram of the foundation chain graph as in Figure 7(e), which has two chain stitches (for the two patches) denoted by two circles in orange and teal, with three foundation chains (for the three crossings), each with a 180-degree twist joining them. Care must be given so that the directions of the twists match with the previous drawing. This is how we set up our Trefoil Knot project. However, to get the same diagram, there is a simpler method introduced in the following section.



(a) *Knot diagram of a right-handed trefoil knot.*

(b) *Deformed into two separate loops.*

(c) *After crossings are put back and arrows removed.*



(d) *A more regular drawing of the surface.*

(e) *Diagram of the foundation chain ribbon graph.*

(f) *Finished foundation chain graph in practice.*

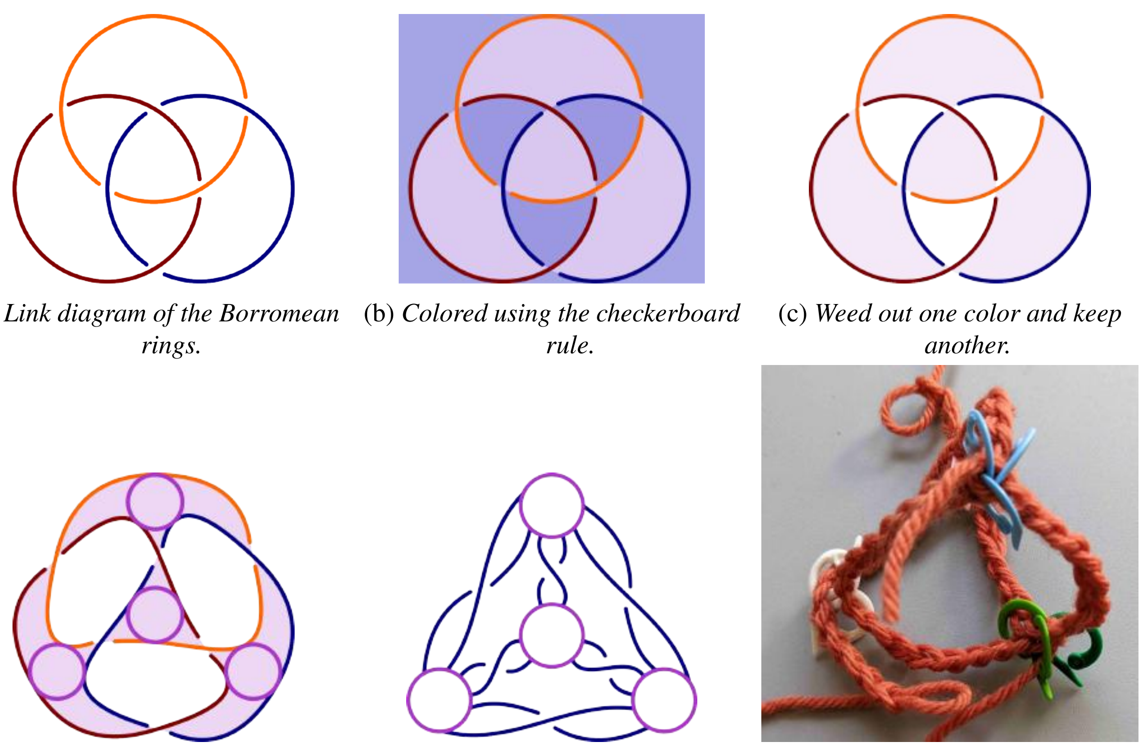
Figure 7: *Application of Seifert's algorithm on the trefoil knot.*

Two-color Algorithm

The key idea underlying Seifert's algorithm is to think of a knot or link diagram as part of a projection of a surface in 3D to 2D. One requirement is that the boundary of the surface needs to be projected onto the knot or link in the diagram. The projection is almost certainly not 1-1, except in highly trivial cases. This leaves many possible ways to "restore" the surface, given its boundary image in 2D. If we abandon the requirement of orientability, a simpler algorithm [1] gives rise to two surfaces from a given knot or link diagram.

The two-color algorithm adapted to topological crochet:

1. Draw the knot or link diagram.
2. Color the whole paper with two colors, with the only constraint that no neighbor regions have the same color. The outer area is also assigned a color. We call it the *checkerboard* rule.
3. Pick one color and collect all the regions with that color, each representing a patch in the surface, which implies a chain stitch. We call this step *weeding*.
4. For each crossing in the original diagram, attach a twisted band joining the patches - the final surface results from all patches with all bands joining them. In topological crochet terms, this means a twisted foundation chain connecting the two corresponding chain stitches.



(a) *Link diagram of the Borromean rings.* (b) *Colored using the checkerboard rule.* (c) *Weed out one color and keep another.*

(d) *A more regular drawing of the surface.* (e) *Diagram of the foundation chain ribbon graph.* (f) *Finished foundation chain graph in practice.*

Figure 8: *Application of two-color algorithm on the Borromean rings.*

Once again, let's see how it works from an example in Figure 8. The final diagram in Figure 8(e) has four chain stitches denoted by four circles in purple, with six bands each with a 180-degree twist joining them. The center of this ribbon graph is the 1-skeleton of a tetrahedron. The reader can check that had we picked the dark purple coloring, it would also been a tetrahedron, but with *opposite* twists for each edge.

Crochet and Sculpture

This workshop series focuses on making yarn sculptures versus more common functional crochet work. Classic sculpture materials are metal, especially bronze, glass, stone, wood, clay, etc. Most of them have intrinsic solidity that allows carving. Some have reasonably reachable melting points, which allows casting and blowing. One unique material is clay, whose temperament largely depends on how moist it is: when wet and soft, it can be easily shaped; when semi-dry, it can be carved. Therefore, soft clay allows for different techniques from other classic materials. One such method is the slab method, in which flat sheets are prepared to be joined and reshaped. The math behind such a method is similar to that of sewing, where premade fabric is cut to be joined. Another method is *coiling*, in which a coil or long pliable cylinder is rolled and placed on top of another. The math behind such a method is similar to that of coiling in basketry. Traditionally, pots, plates, and baskets are made using the coiling method, but they are all topologically very simple.

In topological crochet, we essentially apply the coiling method to yarn to achieve any type of topology instead of just disks and cylinders. There is no absolute algorithm to produce a certain topology—rather, many possibilities exist. Designing and working on topological crochet projects deepens our understanding of topology and provides alternative forms and mediums to the realm of sculpture.

Once one gains enough experience in topological crochet, one can apply the same philosophy to other crafts, such as beading. Naturally, one can also enrich the coiling method in clay and basketry to produce unusual and abstract forms.

Crochet and Topology

A quick review of topological surfaces, and why twist and join are important in making them can be found in [9]. In [10] ribbon graph structures of works of Perry were analyzed. One can use examples in these papers to practice working on different surfaces.

To contemplate what we are really making, the key observation is that crocheting rounds is simply performing the opposite of *deformation retraction* [6]. Such an action constructs a classical *mapping cylinder* M_f from $f : O \to Y$,where in our case O, a one dimensional closed space (hence the choice of letter O), is the knot or link on the boundary of our final piece, Y, almost the same as O except a finite number of branch points (hence the choice of letter Y), is the *skeletal graph* that lies in the center of the foundation chain graph and f is a *degree-2 branched covering*.

To see the latter, very vaguely speaking, any stitch s on the boundary O can be traced back vertically down the crocheted rounds to Y. Such a tracing is very clear in a plain knitted piece. The precise construction here is slightly more complicated, since in crochet a new stitch actually grows in between two old stitches in the lower round and also there will be adding and combining stitches. Nevertheless the mathematical picture remains the same. By construction M_f is a surface (result of degree-2 covering) that is *homotopy equivalent* to Y. The pre-images (or fibers) in M_f of the branch points can be considered as “seams" one could have sewn along had one decided to make M_f in a traditional patch-and-sew way with a differential manifold mindset.

In our current topological method, once we set up the foundation chain graph, we’ve made one “end"(which is not a boundary component) of the mapping cylinder where Y is located and its one narrow neighborhood. All the following rounds of stitches are merely rest of the cylinder part of M_f that is homeomorphic to $O \times I$. We finish crocheting on the other end (the true boundary) of M_f which is O. The beauty of crochet is that the foundation chain rows just have 2 sides to support degree 2 covering, and a single chain can fit as many sides as needed (as in magic ring) to serve as a branch point. Such a perfect connection between crochet and algebraic topology is quite satisfying and mysterious. Hopefully this workshop will be a seed for many new inspiring works to come.

References

[1] C. C. Adams. "Section 2.4: Knots and Planar Graphs." *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*. American Mathematical Society, 2004.

[2] Nat Alison. *Polyhedra Viewer.* https://polyhedra.tessera.li/

[3] C. Bordhi. *MOEBIUS*. https://catbordhi.com/patterns/moebius/

[4] S. Dong. "Sculpting Mapping Cylinders: Seamless Crochet of Topological Surfaces." *Bridges Conference Proceedings*, Halifax, Canada, Jul. 26–31, 2023, pp. 559–566. https://archive.bridgesmathart.org/2023/bridges2023-559.html

[5] E. Eckman. *Crochet, Knit and Learn with Me.* https://www.edieeckman.com/

[6] A. Hatcher. *Algebraic Topology*. Cambridge University Press. https://pi.math.cornell.edu/~hatcher/AT/AT.pdf

[7] F. Herr. *Crocheting a Seifert surface | a math-y crochet project.* https://www.youtube.com/watch?v=UgoGGRhlhPU

[8] C. Livingston and A. H. Moore. *KnotInfo: Table of Knot Invariants.* https://knotinfo.math.indiana.edu/

[9] C. H. Séquin. "2-Manifold Sculptures." *Bridges Conference Proceedings*, Baltimore, Maryland, USA, Jul. 29–Aug. 1, 2015, pp. 17–26. http://archive.bridgesmathart.org/2015/bridges2015-17.html

[10] C. H. Séquin. "Homage to Charles O. Perry." *Bridges Conference Proceedings*, Stockholm, Sweden, Jul. 25–29, 2018, pp. 123–130. http://archive.bridgesmathart.org/2018/bridges2018-123.html

[11] J. J. van Wijk. *Visualization of Seifert Surfaces*. https://www.win.tue.nl/~vanwijk/seifertview/