Counting

In this class, we'll ask *How many ways are there to ...* and discover these kinds of questions are not so easy to answer, at least not easy to answer correctly. It's fine to use a calculator for help; we'll have enough other issues than the arithmetic. In fact, the main difficulty we'll face is keeping straight what exactly it is that we are trying to count!

Fortunately, there is a straightforward and fundamental principle to guide us:

To count a set, build it.

In other words if you put a collection of things together, you can count them along the way. There are two kinds of building steps to count:

Choices multiply. ***Cases add.***

These principles are fundamental, and other tools will be built on top of them. Here are some exercises to show how they apply. Let's count!

1. At the Sooper Scooper Ice Cream Shoppe they will serve you a scoop of Strawberry, a scoop of Chocolate and a scoop of Vanilla ice cream, but the scoops will be stacked up in a random order. If you can guess the order ahead of time, you get the ice cream and cone for free!

   List out the ways to stack the scoops and work out your chances!

2. At the Dippity Dip Ice Cream Store, they have the same three flavors. Their Random Special Cone is two scoops chosen at random—they might be the same flavor or they might be different. How many different special cones are there?

3. Four friends are taking a selfie but can't figure out how they should be arranged. They decide to try every possibility! How many ways do they have to arrange themselves?

4. Uh oh, Here comes a fifth friend. How many ways can they be arranged?

5. Xorxes put its favorite space monster songs on shuffle: "Arrrrgghh," "BBbbburble," "Cdawjhk," and "Dance with me." What are the chances that they are played in alphabetic order?

6. What if Xorxes adds its childhood favorite "Eh" to the shuffle? What is the probability the five songs are played in alphabetical order?

7. If you select five different coins at random, what are the chances that they'll be chosen in order of size?

8. You may have any or all or none of four condiments, your choice of three kinds of cheese, your choice of four kinds of bread, pickle or no pickle, lettuce or no lettuce. How many different sandwiches are possible?

   (How could you get a computer to list these options all out, mechanically?)

9. The fancy ice cream store *Gelatomania* offers three types of container (cone, cup, waffle cone),

three chocolate dips for the container (none, dark, milk), eighty-seven gourmet flavors of ice cream, and optional toppings (nuts, sprinkles, whipped cream and gold dust, each of which may be chosen separately or together from the others), a choice of one of six flavors of chocolate wafer to top it all off, and three kinds of bag to put the ice cream into.

How many options total are there for a family of four people to order ice cream, each with their own idiosyncratic choice of container, dips, two flavors, optional toppings, flavor of wafer and bag to put it all into?

10. There are three ways to walk from `A`'s house to `B`'s house, two ways from `B`'s house to `C`'s, and seven ways from `C`'s directly back to `A`'s house. How many routes pass by `A`'s house, then `B`'s house then `C`'s, finally returning back to `A`'s house?

11. How many of the integers from 1 to 999 have all of their digits odd? (What would be a mechanical way to build up such an integer?)

12. How many of the integers from 1 to 999 have their digits all odd and different from one another?

13. Thirty (different) books sit upon a bookshelf. How many ways are there to select and arrange ten of them on another shelf? $1.09\ 10^{14}$

14. How many six letter strings can be formed from the letters `A B C D E` (possibly using a letter more than once)? 15625

15. How many ways are there to arrange the letters `A B C D E` (using each letter exactly once)? 120

16. How many ways are there to arrange the letters `A B C D E` so that `AE` appears? 24

17. *Some big numbers:* The color of a pixel is specified by three bytes[1], each with eight possible values, for red, green and blue.

    a) How many colors can a pixel be? 16777216

    b) How many possible images[2] are there on a 1280×1024 pixel screen?

    c) This is much greater than the number of subatomic particles in the universe (there are only about 10^{90}).
    How large can a screen be and have fewer possible images than the number of subatomic particles in the universe?

    d) On the other hand, a subatomic particle confined to a handy cubic meter box might be in any of, let's just say, 10^{50} discernible states. How many subatomic particles do you need in the box so that together they have more states than there are possible images on a 1280×1024 pixel screen?

    e) About how many states may a three-dimensional universe be in, measuring the position and velocity of each of 10^{80} particles, within a cube 10^{26} meters on each side, to within 10^{-33} meters of precision?

       (How does this compare to famous numbers such as a googol, googolplex, Graham's number, or the first transfinite ordinal?)

[1] For each of which there are 2^8 possibilities.
[2] Though of course there are vastly fewer humanly distinguishable images! (A less well-defined number.)

Binomials and Multinomials

Dividing a collection of objects into groups of specified sizes is one of the most important building blocks in counting a set. We take this up more completely in Chapter 3 of the main notes. Here are several exercises that show how this is applied.

1. Let's meet each other!

   If we all meet each other how many meetings will we have?

   To answer this, we will try out some simpler questions first:

   (a) If there are just two friends, there is one meeting.
   (b) How many meetings are there for three friends?
   (c) How many meetings for four friends?
   How can we draw a picture of this?
   (d) How many for five friends?
   (e) Is there a pattern?

1. With a little trouble, we can count out, among all sequences of four `H`s or `T`s, exactly how many there are with

   - no `H`s and four `T`s?
   - one `H` and three `T`s?
   - two `H`s and two `T`s?
   - three `H`s and one `T`?
   - four `H`s and no `T`s?

2. Repeat the same exercise for sequences of five coin tosses, six and beyond, and for that matter, three tosses, two or one. What patterns do you find?

3. What is the likelihood that in a random sequence of twenty coin tosses, exactly ten will be `H`s?

(This is the same as asking how many ways are there to order a sequence of ten `H`s and ten `T`s.)

4. How many ways are there to split twenty kids into two groups of ten kids each?

5. How many different orders are there to list out the letters `A S T O U N D`?

6. How many ways are there to match seven kids with seven different adventure opportunities?

7. How many different orders are there to list out the letters `A B R A C A D A B R A`?

8. How many ways are there for a team of eleven kids divide up into different roles: five A's, two B's and R's, one D and one C?

9. Fifteen pixels are each colored, at random with equal likelihood, one of red, green or blue. What is the probability that exactly five pixels are colored red, five are green, and five blue?

10. Fifteen kids are evenly sorted into three teams, a red team, a green team and a blue team. How many arrangements are possible?

Are these games fair?

Here are some games. To understand whether or not they are fair, we need to count out the possibilities within the game.

Game 1

A fair five sided spinner is spun three times.
Want to bet even money that at least once it will come up 1?

Game 2

Let's flip four fair coins. For even money, want to bet there are exactly two H's and two T's? Or bet that there won't be?

Game 3

Roll three dice. If one 6 appears, you get a dollar. If two 6's appear, you get $2. If three 6's appear, you get three dollars! If no 6 appears, you have to pay $1.

The probability that each die shows a 6 is $\frac{1}{6}$, and there are three dice. Half the time, three-sixths, at least one 6 will show, for *at least $1*, so on average you're *sure* to make money*!!* Right?

Game 4

Let's put up a big prize. What about if you pay $1 and roll three dice. If one 6 appears you get your dollar back. If two 6's appear, you get $5 back; if three 6's appear, you get a crisp $50 dollar bill! Sound good? Let's play!

Are these games fair?

Partitions

"Partitions" are a classical counting challenge. How many ways are there to divide up a given counting number into other counting numbers? Equivalently, how many ways are there to divide up a heap of identical objects?

For example, there are eleven partitions of 6:

- 6
- $5 + 1$
- $4 + 2$
- $4 + 1 + 1$
- $3 + 3$
- $3 + 2 + 1$
- $3 + 1 + 1 + 1$
- $2 + 2 + 2$
- $2 + 2 + 1 + 1$
- $2 + 1 + 1 + 1 + 1$
- $1 + 1 + 1 + 1 + 1 + 1$

In some of these, all of the summands are odd (these are the "odd partitions"). In some of these, each summand is unique (these are the "unique partitions"). How many of each kind of partition are there?

How many partitions of 7 are there? List them out! How many odd partitions are there and how many unique ones. Do you see any pattern?

How many partitions of 12 are there? Can you find a formula?

What's going on?

Counting with symmetry

How many ways are there to color parts of a thing, up to its symmetry? For example,

Below are more examples for you to play with. We will learn a clever theory that covers these and beyond.

1. How many ways are there to color:

   (a) the vertices of a rectangle with two W, two G. 3

   (b) the vertices of a square with one R, two G, one B. 2

   (c) the vertices of a regular pentagon with two R, two G, one B. 4

   (d) the eight vertices of a cube with four W, four G. 7

   (e) the eight vertices of a cube with two W, six G. 3

   (f) the six faces of a cube with any three R, three Y. 2

   (g) the twelve edges of a cube with four each of three colors. 1479

2. In these problems, any number of each kind of color can be used – including none! For example, up to symmetry, there are six ways to color the corners of a square with two colors, say R and W: all R, all W; one R three W and *vice versa*; and there are two different ways to color with two R and two W. These numbers are larger than the ones from the problems above, but we'll learn a quicker method to find them. How many ways are there to color:

   (a) the vertices of an equilateral triangle with any of three colors? 10

   (b) the vertices of a square with any of three colors? 21

   (c) the vertices of a regular pentagon with any of three colors? 39

   (d) the vertices of a cube with any of three colors? 333

   (e) the faces of a cube with any of three colors? 57

   (f) the edges of a cube with any of three colors? 22815

Expanding strings

Here's another strange counting example. Consider binary strings made in the following way:

Starting with `1`, successively replace each `1` with `10` and each `0` with `1`. Let's see what we get:

`1`$\rightarrow$ `10`$\rightarrow$ `101`$\rightarrow$ `10110`$\rightarrow$ `10110101`$\rightarrow$ `1011010110110`$\rightarrow \ldots$

How many `1`'s and `0`'s are there at each step?

How fast do these strings grow?

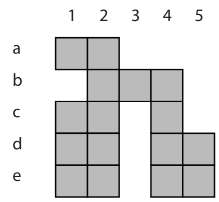
About how many `1`'s and `0`'s will there be after 20 steps?

Counting with restrictions

Suppose we are pairing up items in $\{A, B, C, D, E\}$ to items in $\{1, 2, 3, 4, 5\}$, but subject to some weird complicated conditions. To represent this diagramatically:

We can only match		
A	to	1 or 2
B	to	$2, 3$ or 4
C	to	$1, 2,$ or 4
D	to	$1, 2, 4,$ or 5
E	to	$1, 2, 4,$ or 5

we might draw



These are perfectly reasonable questions:

- Can we match each of A, B, C, D and E with $1, 2, 3, 4,$ and 5, satisfying these conditions? In other words, can we fit 5 rooks into this board?
- If we can, in how many ways may we do this?
- Or to change it up: How many ways may we fit 3 rooks in, with none on the same row or column as any other?
- How can we systemize this?

Another Example

Three inputs XYZ must link to one of four outputs $STUV$. No output may be linked to from more than one input. X may link to STU, Y to TUV, and Z to SV. How many ways may two of the three inputs be linked?

1. Draw a board representing this problem.

Another Example

Six teams A, B, C, D, E, F are available to be paired with four clients, W, X, Y, Z (Two teams will not be needed.) Teams A, B, C are experts on the needs of W, X, Y. Teams D, E work well with

Counting

These notes were originally prepared for a university-level course on combinatorics, and include many examples to try out and learn from. Play around and experiment!

The answers are listed in gray in the margin to check from.

1 How to get started

1. Decks, cards, die rolls and strings are standard combinatorial gadgets. Let's count them!

   Some of these are trickier than others, but just a few ideas work for them all. Look these over, try to answer them as best you can, and think about:

   What principles are we using?

   (a) There are 52 different cards in a standard deck.
   How many ways are there to put four cards in a row? 6497400

   (b) How many ways are there to select four cards together? 270725

   (c) How many ways are there to select four cards together and then put them into order? 6497400

   (d) How many six letter strings may be made of the eight letters `A`, `B`, `C`, `D`, `E`, `F`, `G`, `H`, if letters may be used more than once? 262144

   (e) How many six letter strings may be made of the eight letters `A`, `B`, `C`, `D`, `E`, `F`, `G`, `H`, if letters may be used at most once? 20160

   (f) How many ways can you select six out of the eight letters `A`, `B`, `C`, `D`, `E`, `F`, `G`, `H`? 28

   (g) How many binary strings of length eight have exactly six `0`'s and two `1`'s? 28

   (h) How many binary strings of length eight have at least six `0`'s? 37

   (i) Eight fair six-sided dice are rolled together.
   (Or equivalenty, one die is rolled eight times in a row.)
   What is the probability that all of the rolls show `1` or `2`? 1/6561 $\sim 0.0152\%$
   What is the probability that exactly six of the rolls show `1` or `2`? 112/6561 $\sim 1.71\%$
   What is the probability that at least six of the rolls show `1` or `2`? 43/2187 $\sim 1.97\%$

Before we even get started counting anything, are you sure you know what you're counting?

We will find these principles useful again and again:

Exclusive cases are summed. **Sequential steps are multiplied.**

In a step, if we are choosing one thing out of a pool of n, there are n choices. But more generally, we might have to select k things out of a pool of n, all together, if they are the same in the data. To count these, we use *combinations*, discussed in Section 3.

From a pool of n distinct objects, the number of ways to select k of them is "n choose k", denoted $\binom{n}{k} := (n \cdot (n-1) \dots (n-k+1)) \ / \ k!$

From just these ideas, we will obtain some simple formulas, then build those up.

2. Go back through Exercises 1 and 2, and for each problem, explicitly describe an algorithm that fleshes out the data, and how the number of choices is calculated.

In the next few sections, we'll flesh out these principles, that will be fundamental to the subject of Combinatorics. Later in the semester, we will develop this into more powerful tools: generating functions, the Principle of Inclusion/Exclusion, and Polyá enumeration.

Get some cards, dice, letters, and work through exercises 1, 2 and 3. There's more on the way!

Try out this method on some more subtle problems. Do you get the right answer? Does your algorithm count fairly?

a) Six different books and nine different magazines are available. How many ways can five publications be selected and stacked up if a book must be on top, and a magazine on the bottom? 92664

b) What is the probability that, when five cards from a standard deck are dealt, that the hand has two pairs? 123552/2598960 $\sim 4.75\%$

c) Six different books and nine different magazines are available. How many ways can five publications be selected and stacked up if a book must be on top, and at least one magazine must be in the stack? 143424

d) From a group of three red and four black playing cards, how many ways are there to select four of these so that there is at least one of each color? 34

2 How to count:

2.1 Independent Steps

Each of the following can be counted by multiplying the numbers of cases in a series of independent steps.

1. There are three roads from `A` to `B`, two from `B` to `C`, and seven from `C` to `A`. How many routes are there from `A` to `B` to `C` to `A`? 42
2. You may have any or all or none of four condiments, your choice of three kinds of cheese, your choice of four kinds of bread, pickle or no pickle, lettuce or no lettuce. How many different sandwiches are possible? 768
3. How many possible cars are possible if there are three choices for steering wheel cover, nine choices for...etc...
4. How many routing codes of the form `##X#####-XX` are possible, each `#` one of ten digits `0` through `9`, and each `X` one of the twenty-six capital roman letters `A` through `Z`. 17576000000
5. How many six letter strings can be formed from the letters `A B C D E` (possibly using a letter more than once)? 15625
6. How many ways are there to arrange the letters `A B C D E` (using each letter exactly once)? 120
7. How many ways are there to arrange the letters `A B C D E` so that `AE` appears? 24
8. How many ways are there to arrange beads with the letters `A B C D E` on a necklace? 24
9. How many seven digit passcodes are there using the ten digits `0 - 9`? 10000000
10. How many seven digit passcodes are there using the ten digits `0 - 9`, if no digit may be repeated? 604800
11. Thirty (different) books sit upon a bookshelf. How many ways are there to select and arrange ten of them on another shelf? $1.09\ 10^{14}$
12. Nine students are in the Combinatorics Club. How many ways are there to select a president, vice-president and a secretary? 504

13. In the last three problems, we sought the number of permutations of 7 out of 10 objects, of 10 out of 30 objects, and of 3 out of 9 objects.

    Permutations are counted in consecutive steps of an algorithm: select one of n, then one of $(n-1)$, etc, until there are $(n-k+1)$ choices on the kth and final step.

    In general how many *permutations* — ways to select and distinguish — k out of n objects, are possible?

14. Five CS students and four math students are in the Combinatorics Club. How many ways are there to select a president, a vice-president and a secretary? 504

15. How many ways are there to select a CS student for president, a math student for vice-president and either for secretary? 140

16. How many ways are there to select a president, vice-president and secretary if at least one position must be held by a math student and at least one position must be held by a CS student?[3]

    420, *not* 840. We will soon call this "The Standard Error!".

17. The color of a pixel is specified by three bytes[4], for red, green and blue.

    a) How many colors can a pixel be? 16777216

    b) How many possible images[5] are there on a 1280×1024 pixel screen?

    c) This is much greater than the number of subatomic particles in the universe (there are only about 10^{90}).
    How large can a screen be and have fewer possible images than the number of subatomic particles in the universe?

    d) On the other hand, a subatomic particle confined to a handy cubic meter box might be in any of, let's just say, 10^{50} discernible states. How many subatomic particles do you need in the box so that together they have more states than there are possible images on a 1280×1024 pixel screen?

[3] Does your answer agree with your answer to this question in Section 2.3?
[4] For each of which there are 2^8 possibilities.
[5] Though of course there are vastly fewer humanly distinguishable images! (A less well-defined number.)

2.2 Combinations

"Combinations" — which we count with "binomial coefficients" — are so useful that we'll do a few exercises now and defer more complete discussion to Section 3.

When counting, we count choices at each step of an algorithm to populate a data structure, multiplying from step to step, and (as we'll see in Section 2.3) adding and subtracting our counts as we add or subtract cases. At a step, we can specify one choice out of a pool of n different possibilities (in n ways), but it is also helpful to be able to choose a group of k choices, in $\binom{n}{k} := (n \cdot (n-1) \dots (n-k+1))/k!$, "$n$ choose k" ways.

In this section, we will just begin to use them, but they will prove so helpful they deserve a name: these are *binomial coefficients*.

In Section 3 we will describe why this is the correct number of ways to choose k out of a pool of n.

1. How many ways are there to select four students out of a group of eleven?

   How many ways are there to select seven students out of a group of eleven?

   How many ways are there to divide a group of eleven students into committee A with four students, and committee B with seven? 330

2. How many strings can be formed from four `A`'s and seven `B`'s?

We need more of these, but my imagination is lacking. So as a class, let's make up a few more:

3. How many ways are there to take 11 distinct objects and put 4 of them into bin `A` and seven into bin `B`?
4. How many 10-bit binary strings are there with exactly one `0`? two `0`'s? Three? Four? and on up to... Ten? Make a chart. Check that the total is 2^{10}.

2.3 Cases

Many times it is easier to break a count into cases. If the cases do not overlap, to get the total number, we simply sum them up. If the cases do overlap, well, then it takes some more care. And of course we have to avoid The Standard Error, described in Section 2.3.1.

1. How many possible license plates are there of the form
   (a) `### XXX`, where `#` is a digit `0 - 9` and each `X` is a letter `A - Z`. 17576000
   (b) `### XXX`, or `XXX ###`, or `X#X#X#` 52728000
   (c) `XXX ###`, where none of 164 forbidden three letter strings may appear. 17412000

2.3.1 The Standard Error

2. From two red cards and three black cards, how many ways are there to select a group of four, so that there is at least one of each color? Get out some real playing cards and try this!

   You should be able to find the answer by hand. 5

   But what is wrong with this algorithm: Select one red card (in one of 2 ways). Select one black card (in one of 3 ways). From the remaining three cards, select two, in $\binom{3}{2} := 3{\cdot}2/\,2! = 3$ ways, for a total of $2 \cdot 3 \cdot 3 = 18$ ways. ~~18~~

   What is the correct way to tackle this problem?

In fact this comes up so frequently that we refer to <u>*The Standard Error*</u>, when cases are counted in this particular incorrect way, by mistakingly promoting some object or datum to a special role. In the rest of Section 2.3 and beyond you will see many examples of this.

2.3.2 Accounting for overlapping conditions

3. (a) How many ways are there to arrange the letters `A B C D E` so that `A` appears next to `E`? 48
   (b) How many ways are there to arrange the letters `A B C D E` so that `AE` or `DB` or both appear? 42

(c) How many ways are there to arrange the letters `A B C D E` so that neither `AE` nor `DB` appear?[6] 78

4. Six cards are dealt from a standard deck. How many ways are there to do this

   a) if there are no hearts? 3262623
   b) if there are no hearts and no spades? 230230
   c) if there are no hearts *or* no spades? (so one or the other or neither but not both). 6295016
   d) if there are no `7`'s *and* no `9`'s? 7059052
   e) if there are no `7`'s *or* no `9`'s? 17483972

5. Three fair 10-sided dice are rolled.

   a) What is the probability that there are all even numbered values? $1/8 \sim 12.5\%$
   b) What is the probability that there are at least two numbered values? $99/100 \sim 99\%$
   c) What is the probability that all of the values are 5 or less? $1/8 \sim 12.5\%$
   d) What is the probability that all of the values are even *and* 5 or less? $1/125 \sim 0.8\%$
   e) What is the probability that all of the values are even *or* all of the values are 5 or less? $121/500 \sim 24.2\%$

6. There are five different magazines and six different books.

   (a) How many ways are there to form a stack of four publications if there must be a magazine in the stack? 7560
   (b) How many ways are there to select four publications to give away if one of these must be a magazine? 315
   (c) How many ways are there to stack six publications if there must be a book on top, a magazine on the bottom, and at least one magazine and one book in the middle. 86400

2.3.3 Exclusive Cases

7. Five CS students and four math students are in the Combinatorics Club. How many ways are there to select a president, a vice-president and a secretary

   (a) if the president must be a math student; 224

[6]Problems like these will be a snap, using P.I.E., but for now, try these out directly:

- How many strings can be formed from the thirteen letters `COMBINATORICS`?
- So that the substring `OO` appears? So that the substrings `OO` `CC` and `II` appear?
- So that *none* of the substrings `OO`, `CC` or `II` appear?
- So that *exactly one* of the substrings `OO`, `CC` or `II` appear?
- So that *exactly two* of the substrings `OO`, `CC` or `II` appear?

(b) if exactly one officer is a math student and two are CS students; 240

(c) if exactly two officers are math students and one is a CS student; 180

(d) if at least one officer is a math student and at least one is a CS student.[7] 420

(e) if at least one officer is a math student? 444

(f) Here is an instance of The Standard Error. Why does this *incorrectly* count the number of ways if at least one officer is a math student?

Choose the position for a math student (3 ways) and a student to fill it (4 ways). Then fill the first remaining position with any remaining student (8 ways) and the next remaining position with any student remaining after that (7 ways), or $3 \cdot 4 \cdot 8 \cdot 7 = 672$ *ways.*

(g) How many ways are there to line up six of the students, if there must be a math student on the left end, a CS student on the right end, and at least one of each in the middle positions?

8. There are five math students and six CS students in the Combinatorics Club. How many ways are there to select a committee of four students, if at least one committee member must be a math student, and at least one must be a CS student?

Why is this instance of The Standard Error incorrect: *Choose a math student (in any of* $\binom{6}{1}$ *ways), to choose a CS student (in any of* $\binom{5}{1}$ *ways), and then choose two more students from the remaining nine (in any of* $\binom{5+4}{2}$ *ways), for a total of* $\binom{6}{1}\binom{5}{1}\binom{9}{2}$.

Instead we may sort out the collection of ways of choosing four students from eleven into cases: how many CS students and how many math students were chosen. These cases are easy to count, and can be summed to get the number we want.

How many ways are there to choose:

(a) four students out of eleven? 330

(b) four math students and no CS students? 5

(c) three math students and one CS students? 60

(d) two math students and two CS students? 150

(e) one math students and three CS students? 100

(f) no math students and four CS students? 15

(g) Check that the first of those equals the total of the rest.

(h) What is the number of ways to choose four students, at least one CS major and at least math major? 310

9. How many 10-bit binary strings have at least two `0`'s and two `1`'s? 1002

10. Ten fair six-sided dice are rolled. What is the probability that there are no more than two 6's? $\sim 77.53\%$

11. A row of six cards is dealt from a standard 52-card deck. What is the probability that four or more of the cards are red? $\sim 33.3976\%$

[7] Does your answer agree with your answer to this question in the previous section?

12. How many strings using the letters `a a b b c` are possible if
    a) `aa` must not appear. 18
    b) `ac` and `ca` must not appear. 9
    c) How many n-letter strings using `a`, `b`, `c` are possible if `ac` and `ca` must not appear. (We will have a good approach to this soon, but try it out now.)

In Section 4 we will take up P.I.E., the Principle of Inclusion-Exclusion, which will help us count up more complicated arrangements of conditions.

2.4 More elaborate counts

The things we are counting are specified by their data, and constraints on this data. We *multiply* when we construct our data in a series of independent steps. We *add* when our data is in mutually exclusive cases. Now we combine these in more complex ways. The key is:

What is a data structure encoding what we wish to count?

1. The fifty-two standard playing cards in four suits ♠, ♡, ♣, ♢ and thirteen denominations `A 2 3 4 5 6 7 8 9 10 J Q K`. A "hand" is an unordered selection of five of these fifty-two cards.
    a) How many hands are possible? (What general size of number is this?) 2598960
    b) How many flushes (all the same suit)? (What is the probability a random hand is a flush?) 5148,∼ .198%
    c) How many straights? if `A`'s can be either low or high, 10240
    d) How many hands with five different denominations (no pair of cards have the same denomination)? 1317888
    e) How many worthless hands are there? Let's not worry about straights and ask: How many hands have all unmatched ranks and at least two suits? 1312740
    f) How many hands with a pair and three unmatched cards? 1098240
    g) How many hands with two pair and one unmatched card? 123552
    h) Three of a kind? Full house? Four of a kind? Five? [8]
    i) How many hands of thirteen cards have two pair, two three-of-a-kind and three unmatched cards? $\sim 1.328\ 10^{11}$

[8]For some fun, include a pair of jokers in the enumeration!

2. Standard dice show `1 2 3 4 5 6`. Five dice[9] are thrown.

   a) How many throws are possible? 7776

   b) How many throws with all five dice different (no pair of dice has the same denomination)? 720

   c) How many throws with a pair and three unmatched dice? 3600

   d) How many throws with two pair and one unmatched die? 1800

   e) How many throws of thirteen eight-sided dice have two pair, two three-of-a-kind and three unmatched rolls? 72648576000

3. Consider strings of length 20 of 0's, 1's, 2's and 3's.

   a) How many consist of exactly five 0's, six 1's, seven 2's and the rest 3's? 2793510720

   b) How many consist of six 1's, seven 2's and at least two 0's?

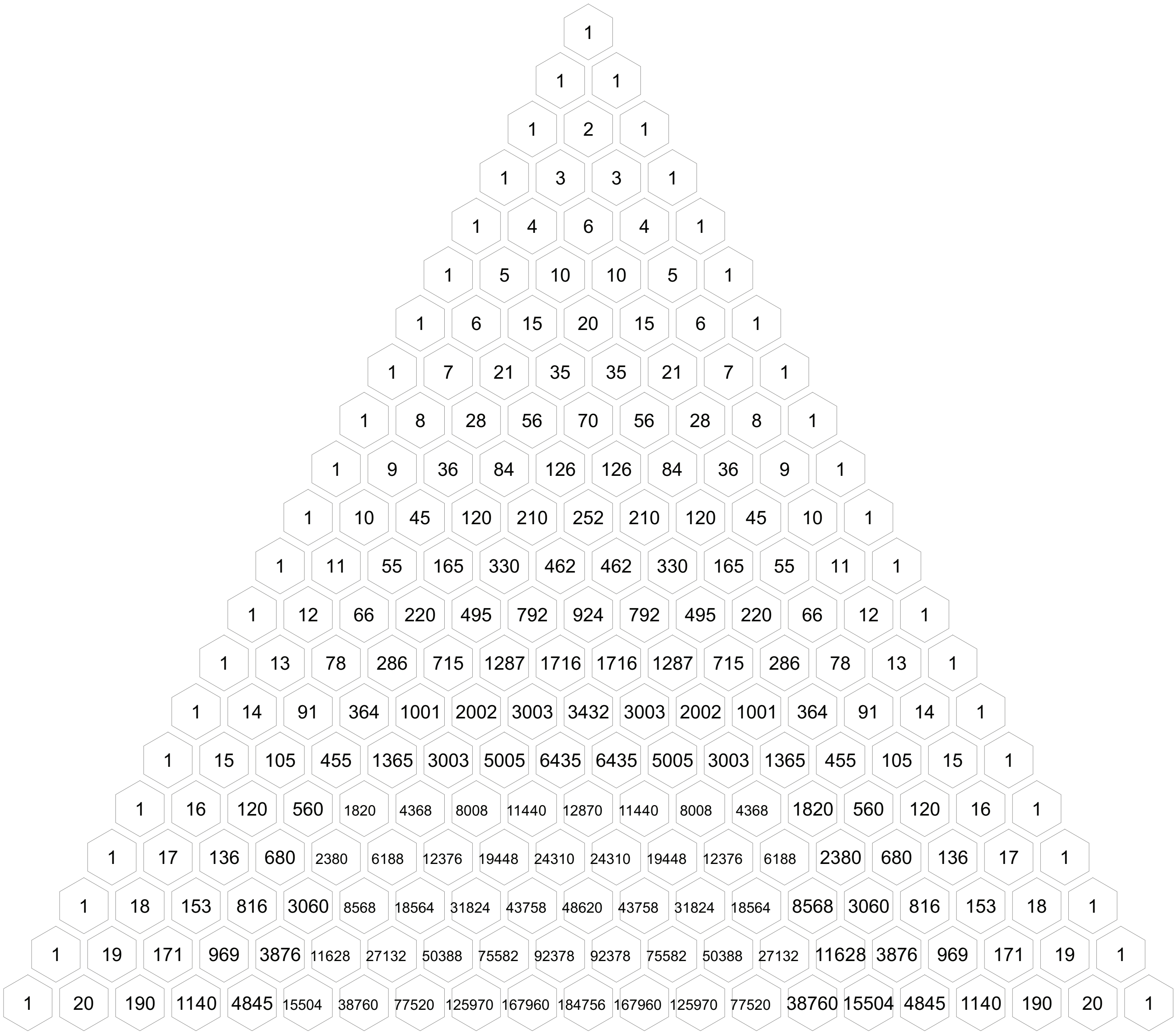
   c) How many consist of at least three 1's, at most seven 2's and at least two 0's?

   d) How many strings of length 20 have "weight" — the sum of digits — at least 6?

[9] These are physical dice, so a `1` on the first die and `2` on the second isn't the same thing at all as a `2` on the first and a `1` on the second.

3 Binomial and Multinomial Coefficients

The numbers in this table appear again and again, remarkably often. They simply record the number of different direct routes from the top of the table to their own position:



3.1 Binomial coefficients

For any positive integer k, we define $k! = k \cdot (k-1) \cdots 2 \cdot 1$, or more succinctly $k! := \prod_{j=1}^{k} j$.

It turns out to be helpful to define $0! := 1$. One way to see that this makes sense is that $2! = 3!/3$ and $1! = 2!/2$. Consequently, $0!$ should be $1!/1 = 1$ — though this breaks down for $(-1)!$ etc. [10]

If k is not a non-negative integer, we define the "binomial coefficient"

$$\binom{n}{k} := 0$$

Otherwise, for any non-negative integer k and any real number n, we define

$$\binom{n}{k} := \frac{(n)(n-1)\cdots(n-k+1)}{k!}$$

For reasons we'll soon discuss, we pronounce this "n choose k".

So for example $\binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10$. Since the 3's cancel, note that $\binom{5}{3} = \binom{5}{2}$, and in general,

For non-negative integers n, j, k with $j + k = n$, we have $\binom{n}{j} = \binom{n}{k} = \frac{n!}{j!k!}$.

But we have defined the binomials even when n is not a non-negative integer! For example $\binom{1/2}{3} = \frac{(1/2)(-1/2)(-3/2)}{3!} = \frac{3}{16}$, and

$$\binom{-3}{4} = \frac{(-3)\cdot(-4)\cdot(-5)\cdot(-6)}{4!} = (-1)^4 \frac{6 \cdot 5 \cdot 4 \cdot 3}{4!} = (-1)^4 \binom{3+4-1}{4} = (-1)^4 \binom{3+4-1}{3-1} = 15$$

In general, check that

if n is a positive integer (and $-n$ is a negative integer) then $\binom{-n}{k} = (-1)^k \binom{n+k-1}{n-1}$.

[10] The "gamma function", $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt$ fills in factorials for all real numbers *except* the negative integers: By the method of integration by parts, one proves that for all x, $\Gamma(x+1) = x\Gamma(x)$. Since $\Gamma(1) = \int_0^\infty e^{-t} = 1$, we have by induction that $\Gamma(n+1) = n!$ for all non-negative integers n. The indefinite integral defining $\Gamma(x)$ does not converge for negative values of x.

Because $0! = 1$, for any non-negative integer n, $\binom{n}{0} = \frac{n!}{n!0!} = 1$. We simply extend this definition to all real numbers n and define $\boxed{\binom{n}{0} := 1}$

$\boxed{\text{For non-negative integers } k, n, \text{ with } k > n, \text{ we have } \binom{n}{k} = 0}$ because $\binom{n}{k} = \frac{n \cdots 0 \cdots}{k!}$.

Quick check. Evaluate:

- $\binom{15}{0}$
- $\binom{6}{4}$
- $\binom{2}{4}$
- $\binom{7}{10}$
- $\binom{2/3}{2}$
- $\binom{12}{1}$
- $\binom{10}{7}$
- $\binom{-12}{1}$
- $\binom{3}{-3}$
- $\binom{\pi}{4}$
- $\binom{10}{3}$
- $\binom{-3}{3}$
- $\binom{-12}{0}$
- $\binom{4}{-2}$
- $\binom{0}{0}$

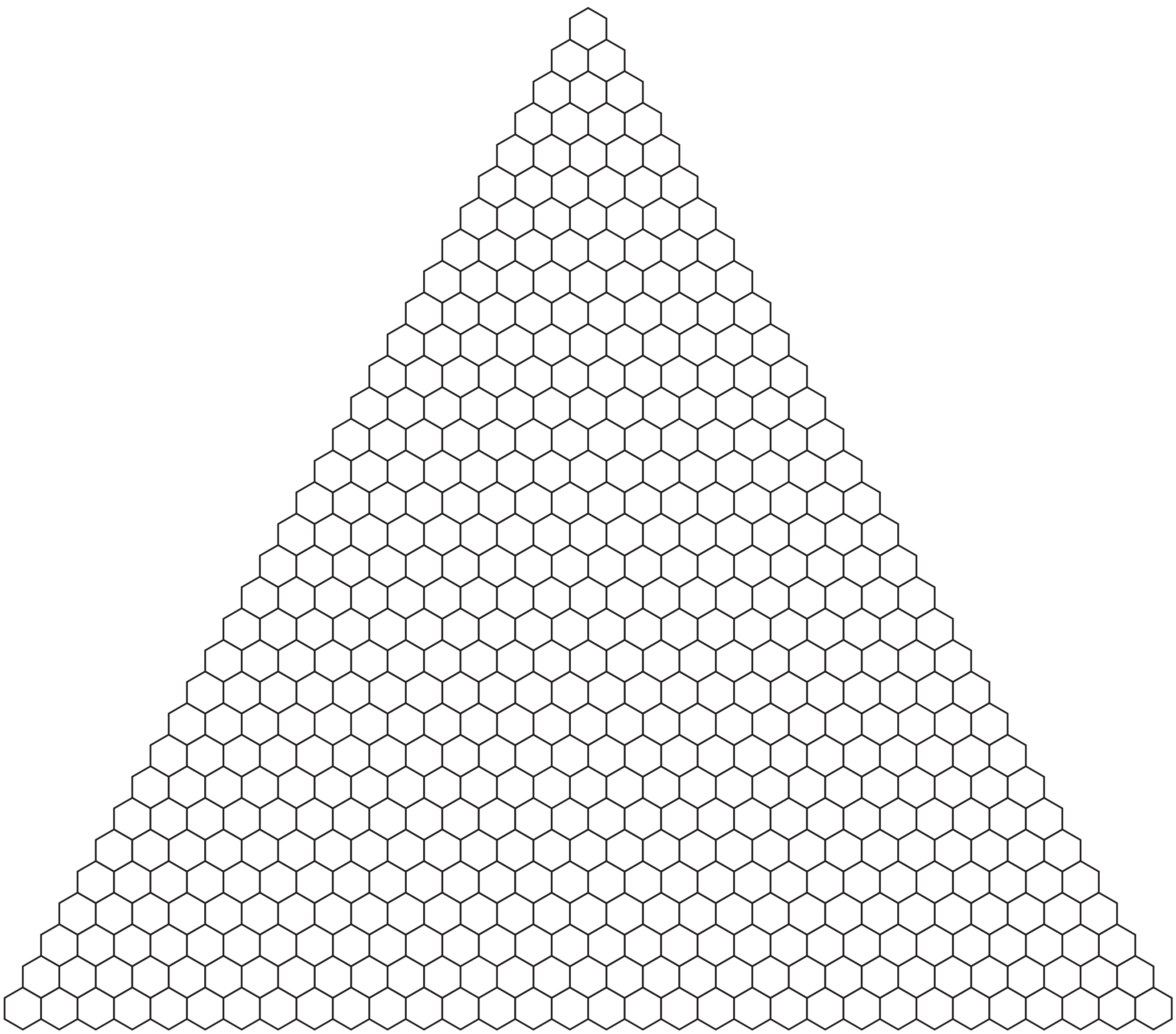
It is worth proving that $\boxed{\text{for } k \neq 0, \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}}$. If k is not a positive integer (and $k \neq 0$), then this is because $0 = 0 + 0$. Otherwise, if k is a positive integer (and $(k-1)$ is therefore non-negative) we apply our definition and have

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)\cdots((n-1)-k+1)}{k!} + \frac{(n-1)\cdots((n-1)-(k-1)+1)}{(k-1)!} \\
&= \frac{(n-1)\cdots(n-k)}{k!} + \frac{(n-1)\cdots(n-k+1)}{(k-1)!} \\
&= (n-k)\left(\frac{(n-1)\cdots(n-k+1)}{k!}\right) + (k)\left(\frac{(n-1)\cdots(n-k+1)}{k!}\right) \\
&= n\left(\frac{(n-1)\cdots(n-k+1)}{k!}\right) \\
&= \frac{n\cdots(n-k+1)}{k!} = \binom{n}{k}
\end{aligned}$$

This gives us the famous triangle in which each entry is the sum of the two above, known to Pascal (1623-1662), Yang Hui 杨辉三角 (1238-1298), Jia Xian 贾宪 (1010-1070) and Al-Karaji ابو بكر محمد بن الحسن الكرجى (c. 953 – c. 1029)

$$\begin{array}{ccccccccccc} & & & & & \binom{0}{0} & & & & & \\ & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\ & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\ & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\ & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\ \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\ & & & & & \textit{etc.} & & & & & \end{array} \quad \text{or} \quad \begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ & & & & & \textit{etc.} & & & & & \end{array}$$

Notice that all the missing entries are for $k < 0$ or $k > n$— we may fill in all the missing entries with 0's and the table will still be correct!



What the binomial coefficients are counting

For positive integers n and any integer k, $\binom{n}{k}$ counts the number of ways to choose k objects from a pool of n objects.

For example $\binom{6}{4}$ counts the fifteen ways to pick four out of the six letters `a`,`b`, `c`, `d`, `e` and `f`. (Try listing these out. Notice that we don't care in which order the letters are selected and the choice `acdf`, for example, is just the same as `dafc` etc)

In the special case that $k = 0$, recall $\binom{n}{0} = 1$. Since there is exactly one way to choose no objects from a pool of n objects — do nothing!— $\binom{n}{0}$ counts this correctly in this case! And when $k < 0$ or $k > n$ there are no ways to choose k out of n objects, and indeed here, $\binom{n}{k} = 0$.

For positive k let's prove $\binom{n}{k}$ counts the number of ways to choose k out of n objects, in two ways:

First, we'll prove this by a direct counting argument: If we wish to arrange k out of n objects in a line, there are n choices for the first in line. But there are now $(n-1)$ choices for the second in line, leaving $(n-2)$ for the third, etc, until there remain $(n-k+1)$ for the kth in line. So there are $n \cdots (n-k+1)$ ways to arrange k of n objects in a line. But this overcounts each of what we wanted to count many times. Each choice of k objects could have been arranged in a line in $k!$ ways, and so was counted $k!$ times. So the number of choices of k objects from a pool of n is $\dfrac{n \cdots (n-k+1)}{k!} = \binom{n}{k}$.

Secondly, let's prove this by induction as well. For $n = 1$, there is $1 = \binom{1}{0}$ way not to choose a single object and $1 = \binom{1}{1}$ way to choose it.

So assume that for a given $n-1$, for any j, $\binom{n-1}{j}$ correctly counts the number of ways to choose j out of $n-1$ objects. How many ways are there to choose k out of n objects? Of all possible choices, there are two possibilities: Either the nth object is chosen, or not. If the nth object is included in the choice, we must have chosen $(k-1)$ objects from a pool of $(n-1)$. If the nth object is not included, we must have chosen k objects from a pool of n. In short the number of ways to choose k out of n objects is $\binom{n-1}{k-1} + \binom{n-1}{k}$, which we have already proven equal to $\binom{n}{k}$.

Equivalently

For non-negative integers j, k and arbitrary symbols `A`, `B`, $\binom{j+k}{j} = \binom{j+k}{k}$ counts the number of strings with j `A`'s and k `B`'s.

This is simply because we must choose j out of the $j+k$ positions in the string where the `A`'s will appear, or equivalently, the k positions where the `B`'s appear.

Equivalently

For non-negative integers j, k $\binom{j+k}{j} = \binom{j+k}{k}$ counts the number of ways to partition $i+j$ identical objects into two distinct categories, the first with i objects and the second with j objects.

Binomial coefficients in the binomial theorem

When we multiply $(a+b)(y+z) = ay + az + by + bz$, our terms correspond to the ways of choosing one of a and b from the first multiplicand, multiplied by one of y and z from the second (A.k.a. "FOIL"). This generalizes fully: when we multiply out any product of sums, our terms will exactly correspond to the ways of choosing one summand from each of the multiplicands.

In particular, how do we multiply out $(a+b)(a+b)(a+b)$?

There are eight terms, corresponding to the two ways to choose an a or b from the first $(a+b)$, times the two ways to choose an a or b from the third $(a+b)$.

$$(a+b)(a+b)(a+b) = aaa + aab + aba + abb + baa + bab + bba + bbb$$

This simplifies to

$$(a+b)(a+b)(a+b) = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

Where did the coefficients come from? The coefficient 3 of a^2b arose because $3a^2b = aab + aba + baa$ — there are three different ways to string together two a's and one b.

In general, for non-negative integers j, k, n with $j+k=n$, the coefficient of $a^j b^k$ in $(a+b)^n$ will be exactly the number of ways to string together j a's and k b's– that is, exactly $\binom{n}{j} = \binom{n}{k}$ and we have the binomial theorem in this case: [11]

For counting number n,

$$(a+b)^n = \binom{n}{0} a^0 b^n + \binom{n}{1} a^1 b^{n-1} + \ldots + \binom{n}{1} a^{n-1} b^1 + \binom{n}{0} a^n b^0$$

In other words, the coefficients are the corresponding row of Pascal's triangle.

In fact, remembering that $\binom{n}{k} = 0$ for counting numbers n, k with $n < k$, we actually have that

$$(a+b)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^k b^{n-k}$$

But[12] this actually holds *for all* real numbers n! To be more precise,

For all real numbers n,

$$(1+x)^n = \binom{n}{0} x^0 + \binom{n}{1} x^1 + \cdots = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

[11]Here is a more rigorous proof by induction on n: For $n=1$, $(a+b)^1 = \binom{1}{0} a^0 b^1 + \binom{1}{1} a^1 b^0$. Suppose the theorem holds for a given $n-1$. Then $(a+b)^n = (a+b)^{n-1}(a+b) = a(a+b)^{n-1} + b(a+b)^{n-1}$. On the left hand side, the coefficient of $a^j b^k$, with $j+k=n$, is the sum of the coefficient of $a^{j-1} b^k$ in $(a+b)^{n-1}$ and the coefficient of $a^j b^{k-1}$ in $(a+b)^{n-1}$, in other words, $\binom{n-1}{j-1} + \binom{n-1}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ which we have seen equals $\binom{n}{k}$.

[12]The general statement is a consequence of Taylor's Theorem, for a function $f(x)$, if at some a, $f(a), f'(a), f''(a), ..., f^{(k)}(a), ...$ are all defined then in a neighborhood of a, $f(x) = \sum_{k=0}^{\infty} f^{(k)}(a)(x-a)^k/k!$

In other words,

> The coefficients of each x^k in $(1+x)^n$ are $\binom{n}{k}$. The function $(1+x)^n$ is the **generating function** for the sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots$.

Exercises

1. Let's derive some binomial coefficients from scratch:

   (a) How many ways are there to arrange the letters `A B C D E`?

   (b) How many ways are there to arrange the (all different) letters $\boldsymbol{A}$ $\mathbb{A}$ $\mathcal{A}$ e E?

   (c) Of these, how many were of the form `E A E A A`? How many of the form `A A E A E`? etc?

   (d) In general, for any arrangement of the letters `A A A E E`
   how many corresponding arrangements are there of the letters $\boldsymbol{A}$ $\mathbb{A}$ $\mathcal{A}$ e E ?

   (e) How many arrangements are there of the letters `A A A E E` ?

   (f) In general, how many arrangements are there of i `A`'s and j `B`'s?

2. For any $n = i + j$, for $n, i, j \geq 0$, why are the following all equivalent?

   - The number of ways to choose, out of n things, i of them;
   - The number of ways to choose, out of n things, $n - i = j$ of them;
   - The number of ways to arrange i `A`'s and j `B`'s;
   - The number of ways to distribute n objects, i into the `A` pile and $n - i$ in the `B` pile;
   - (less obviously) the coefficient of $a^i b^j$ in $(a+b)^n$;
   - (less obviously) in Pascal's triangle, the ith entry in the nth row (counting both from 0)
   - $\binom{n}{i} := n \cdot (n-1) \cdot \cdots \cdot (n-i+1) \ / \ i!$, with i multiplicands in both the numerator and denominator.
   - $\binom{n}{j}$

3. (a) How many ways are there to select four students out of a group of eleven? 330
   How many ways are there to select seven students out of a group of eleven? 330

   (b) How many strings can be formed from four `A`'s and seven `B`'s? 330

(c) How many ways are there to take 11 files and put 4 of them into trashcan `A` and seven into trashcan `B`? 330

(d) Make up some of your own!

4. Prove that for a fixed counting number n, $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$.
(You can also prove this with a simple counting argument.)

5. Similarly prove that $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n} = 0$.
This is trivial for odd n (why) but kind of amazing for even n.

Coefficients

Let's calculate actual coefficients: 35

6. What are the coefficients of a^4b^3 in -560

- a^4b^3 in $(a+b)^7$
- a^4b^3 in $(2a-b)^7$
- a^ib^j in $(2a-b)^7$, for arbitrary $i, j \in \mathbb{Z}$?

Often we will write $[x^n]f(x)$ to mean the coefficient of x^n in the expansion of $f(x)$. For example, the binomial theorem can be stated as:

$$[x^n](1+x)^k = \binom{k}{n}$$

(Are k and n correct? Check: What is $[x^3](1+x)^6$?)

We have some rules (which are obvious if you think about them) such as

- $[x^n](f+g) = ([x^n]f) + ([x^n]g)$
- $[x^n](cf+g) = c[x^n]f$

for functions f, g and constant c. There's another less obvious but very helpful rule. Why is this true?

$$\bullet [x^n]xf = [x^{n-1}]f$$

So for example, the coefficient of x^4 in $x(1-2x)^{10}$ is:

$$[x^4]x(1-2x)^{10} = [x^3](1-2x)^{10} = (-2)^3\binom{10}{3}$$

7. What are the coefficients of x^5 in 312741

13608

$-1701/256$

~ 1.8904

40095

40501

- $(1-3x)^9$
- $1/(1-3x)^4$
- $(1-3x)^{1/2}$
- $(1-3x)^\pi$
- $1/(2+x)^4$
- $1/(2+3x)^5$
- $x(1+2x)^5$
- $x/(1-3x)^9$
- $(x+x^3+x^5)/(1-3x)^9$

3.2 Multinomial coefficients

We will also need multinomials:

1. (a) How many ways are there to arrange the eleven letters A to K?[13]
   (b) How many ways are there to arrange the (all different) letters $\mathtt{M}\ \mathcal{I}\ \mathbb{S}\ s\ \mathbb{I}\ \mathbf{s}\ \mathcal{S}\ \mathbf{i}\ \rho\ P\ \mathsf{I}$
   (c) Of these, how many were of the form M I S S I S S I P P I?
   (d) In general, for any arrangement of one M, four I's, four S's and two P's, how many arrangements are there of the eleven letters M, $\mathcal{I}\ \mathbb{I}\ \mathbf{i}\ \mathsf{I}$, $\mathbb{S}\ s\ \mathbf{s}\ \mathcal{S}$, $\rho\ P$?
   (e) How many arrangements are there of the letters one M, four I's, four S's and two P's ?
   (f) In general, how many arrangements are there of m M's, i I's, s S's and p P's
   (g) For $n = m+i+s+p$, each $m, i, s, p \geq 0$, what is the coefficient of $\mathtt{M}^m\mathtt{I}^i\mathtt{S}^s\mathtt{P}^p$ in $(\mathtt{M}+\mathtt{I}+\mathtt{S}+\mathtt{P})^n$?

Multinomial coefficients are useful for counting the number of ways to distribute distinct objects into distinct bins:

2. Sixteen students will be assigned to the Red, Green, Blue and Yellow teams, four to each team. In how many ways may this be done? 63063000
3. Twelve different books will be distributed equally to four children.
   How many ways may this be done? 369600
4. Fifteen job requests are equally distributed to five distinct work units.
   How many ways may this be done? 168168000
5. Six managers read 24 complaints.
   a) How many ways can the complaints be distributed so each manager reads four? $\sim 3.247\ 10^{15}$
   b) How many ways can the complaints be distributed so that managers A and B read six complaints each and managers C D E F each read three? $\sim 9.235\ 10^{14}$

c) How many ways can the complaints be distributed so that two managers each read six and four managers each read three? $\sim 1.385\ 10^{16}$

6. Six packets are each randomly assigned to one of three routers. What is the probability that each router has been assigned exactly two packets? 10/81

7. We have seen that the number of strings of a `A`'s and b `B`'s is $\binom{a+b}{a} = \binom{a+b}{b} = \frac{(a+b)!}{a!\ b!}$.

   a) Why is the number of strings of a `A`'s, b `B`'s, and c `C`'s exactly the "multinomial coefficient" $\frac{(a+b+c)!}{a!\ b!\ c!}$?

   b) What is the number of strings of a `A`'s, $\ldots k$ `K`'s?

   c) How many ways are there to arrange the letters `A B R A C A D A B R A`? 83160

   d) What is the coefficient of $\mathtt{A}^5\mathtt{B}^2\mathtt{C}^1\mathtt{D}^1\mathtt{R}^2$ in the expansion of $(\mathtt{A}+\mathtt{B}+\mathtt{C}+\mathtt{D}+\mathtt{R})^{11}$? (The number of arrangements of `ABRACADABRA`). 83160

   e) What is the coefficient of $\mathtt{A}^5\mathtt{B}^2\mathtt{C}^1\mathtt{D}^1\mathtt{R}^2$ in the expansion of $(\mathtt{A}-\mathtt{B}+\mathtt{C}-\mathtt{D}+2\mathtt{R})^{11}$? -332640

   f) What is the coefficient of $\mathtt{A}^5\mathtt{B}^2\mathtt{C}^1\mathtt{D}^1\mathtt{R}^2$ in the expansion of $(\mathtt{A}-2\mathtt{B}+3\mathtt{C}-4\mathtt{D}+5\mathtt{R})^{11}$? -99792000

   g) State and prove a theorem about the coefficients in the expansion of $(\mathtt{A}+\mathtt{B}+\cdots+\mathtt{K})^n$.

   h) Prove that $k^n = \sum_{n=\sum^k c_i} \frac{n!}{c_1!\ c_2! \ldots c_k!}$

8. (a) In how many ways can 15 players form into `Red`, `Green` and `Blue` teams, of 5 players each?

   (b) In how many ways can 15 players form into three teams, of 5 players each?

3.3 A useful thing

How many terms are there in the expansion of $(\mathtt{A}+\mathtt{B}+\mathtt{C}+\cdots+k)^n$? For example,

$$\begin{aligned}(\mathtt{A}+\mathtt{B}+\mathtt{C})^5 = \ & 30\mathtt{A}^2\mathtt{B}^2\mathtt{C} + 10\mathtt{A}^3\mathtt{B}^2 + 10\mathtt{A}^2\mathtt{B}^3 + 30\mathtt{A}^2\mathtt{B}\mathtt{C}^2 + 20\mathtt{A}^3\mathtt{B}\mathtt{C} + 5\mathtt{A}^4\mathtt{B} + 10\mathtt{A}^3\mathtt{C}^2 + 10\mathtt{A}^2\mathtt{C}^3 + 5\mathtt{A}^4\mathtt{C} + \mathtt{A}^5 \\ & +30\mathtt{A}\mathtt{B}^2\mathtt{C}^2 + 20\mathtt{A}\mathtt{B}^3\mathtt{C} + 5\mathtt{A}\mathtt{B}^4 + 20\mathtt{A}\mathtt{B}\mathtt{C}^3 + 5\mathtt{A}\mathtt{C}^4 + 10\mathtt{B}^2\mathtt{C}^3 + 10\mathtt{B}^3\mathtt{C}^2 + 5\mathtt{B}^4\mathtt{C} + \mathtt{B}^5 + 5\mathtt{B}\mathtt{C}^4 + \mathtt{C}^5\end{aligned}$$

and you can count out that there are 21 terms.

Each term is of the form $\mathtt{A}^a\mathtt{B}^b\mathtt{C}^c$ where a, b, c are • integers, • non-negative, and • sum to 5.
(And we know the coefficient: $\frac{5!}{a!b!c!}$)

[13] More generally, for any $k \geq 1$, how many ways are there to arrange `A` through the kth letter `K`?

In other words there are exactly as many terms as there are solutions to

$$a+b+c=5, \qquad 0 \le a,b,c$$

Here in this section, we'll count these in one way, and see that there are $\binom{5+(3-1)}{(3-1)} = 21$ solutions, or 21 terms, or 21 ways to put 5 identical objects into 3 distinct bins.

In Section 5 we'll have another approach, and count this as $(-1)^5\binom{-3}{5} = 21$, the coefficient of x^5 in the expansion of $(1-x)^{-3}$. (You'll see!) The method there will be the most general.

This kind of system comes up naturally when we are distributing identical objects into distinct bins. For example

How many ways are there to distribute 6 identical things into bins `A`, `B`, `C`, `D`? If we put a things into bin `A`, b things in to `B`, c into `C`, and d things into bin `D`, we are asking how many integer solutions are there to the equation

$$a+b+c+d=6$$

requiring that

$$a,b,c,d \ge 0$$

For $n=6$, try to work the answer out by hand! 84

Here is a clever trick: Any solution to $a+b+c+d=n$ can be represented as a string of n $*$'s and three $|$'s. For example, with $n=6$, the solution $a=3, b=0, c=1, d=2$ is encoded as $\underbrace{*\ *\ *}_{a=3} \mid \underbrace{\quad}_{b=0} \mid \underbrace{*}_{c=1} \mid \underbrace{*\ *}_{d=2}$

- Encode the solution $a=2, b=1, c=1, d=2$ to $a+b+c+d=6$ as a string.
- What solution does the string `| | * * * * | * *` encode?

The key is that there is a *bijection*, an <u>exact one-to-one and onto correspondence</u> between the strings of six $*$'s, $(4-1)$ `|`'s, and the non-negative integer solutions to the equation $a+b+c+d=6$, partitioning 6 identical objects into four distinct groups.

More generally:

The number of ways to distribute n identical objects into k distinct groups is $\binom{n+(k-1)}{k-1}$

Equivalently, this is the number of solutions to

$$n = a + b + \cdots + k, \quad \text{with } a, b, \ldots, k \geq 0$$

Equivalently, this is the number of terms in the expansion of

$$(A + B + \cdots + K)^n = \sum_{a+b+\cdots+k=n} \frac{n!}{a!b!..k!}\ A^a B^b..K^k$$

Exercises

1. How many ways are there to distribute 17 identical cookies to `Merinda` `Ira` `Sally` `Paulo`? 1140
2. In how many ways may nine identical bowls be thrown into any of three drawers? 55
3. In how many ways may six identical monitors be distributed among eleven work stations (not everyone gets one!) 8008
4. How many terms are in the expansion of the polynomial $(\mathtt{M} + \mathtt{I} + \mathtt{S} + \mathtt{P})^n$? Of $(a + b + c)^9$? ....,55
5. How many ways are there to
   (a) distribute n identical objects into k different bins;
   (b) label n objects with k labels ;
   (c) solve $n = \mathtt{a} + \cdots + \mathtt{k}$, each $\mathtt{a} \ldots \mathtt{k} \geq 0$;
   (d) write n as the ordered sum of k non-negative summands?
   (e) What is the coefficient of x^n in $(1 - x)^{-k}$?
6. We can expand on this method, changing the *lower* bounds on each variable:
   (a) Why is the number of solutions to

$$a + b + c + d = 10 \ \text{ with } a \geq 2, b \geq 1, c, d \geq 0$$

the same as the number of solutions to

$$a + b + c + d = 7 \text{ with } a, b, c, d \geq 0$$

120

Solve the following, noticing that any combination of lower bounds can be handled easily:

(b) $a + b + c + d + e = 20$ with $a, b, c, d, e \geq 0$. 10626

(c) $a + b + c + d + e = 20$ with $a, b, c, d, e \geq 3$. 126

(d) $a + b + c + d + e = 20$ with $a \geq -2, b \geq -1, c \geq 0, d \geq 1, e \geq 2$. 10626

(e) $w + x + y + z = 30$ with $0 \leq w$, $2 \leq x$, $-2 \leq y$, $-4 \leq z$ 7770

7. How many way are there to give 20 identical candies to four children if each child must have at least three candies? 165

8. How many way are there to distribute 20 identical candies to Alice, Bob, Cleo and Dina if Alice must have at least seven, Bob at least five, Cleo at least three and Dina at least one? 35

9. How many ways are there to give twenty identical unicorn stickers to four kids, if each kid must have at least three? 165

10. How many ways to distribute 20 identical batteries to four technicians if each technician must receive at least three batteries? 165

11. How many ways are there to place 20 identical pieces of waste into four distinct bins if each bin must have at least three pieces of waste? 165

12. Make up and solve a couple more of these.

Upper bounds are more problematic.

We can handle *one* upper bound: The number of solutions to

$$a + b + c = 5, \qquad a, b, c \geq 0, \text{ and } a \leq 3$$

is the *total* number of solutions, with $a, b, c \geq 0$, *minus* the solutions we don't want, those with $a \geq 4$. 18

13. How many solutions are there to

(a) $a + b + c + d = 10$ with $0 \leq a \leq 4$ and $b, c, d \geq 0$? 230

(b) $v + w + x + y + z = 10$ with $0 \leq v \leq 2, 0 \leq w, x, y, z$? 671

(c) $i + j + k = 6$ with $i, j, k \geq 1$, $i \leq 2$? 7

Handling *two* upper bounds is not too bad:

The number of solutions to

$$a+b+c=5, \qquad a,b,c \geq 0, \text{ and } a,b \leq 3$$

is the *total* number of solutions, with $a,b,c \geq 0$, *minus* those with $a \geq 4$ *minus* those with $b \geq 4$, but then *plus* those with $a \geq 4$ *and* $b \geq 4$.

14. How many solutions to $a+b+c+d=9$ with $a,b,c,d \geq 0$ and

- $a \leq 5$
- $a,b \leq 5$
- $a,b,c,d \leq 5$

200,180,140

To handle the complex logic of the last example, we'll turn to the Principle of Inclusion and Exclusion, P.I.E in the next Section 4.

4 The Principle of Inclusion-Exclusion (P.I.E.)

These examples — and we'll see many more — satisfy some conditions that are easy to count, but their negations are harder to handle:

• Forbidden substrings: It is easy to count the number of arrangements of `A B C D E F G H` in which some substrings, say `ACD`, `BG` and `EH` *must* appear. It is less obvious how to count those in which *none* of those substrings appear, in other words how to count those in which *all* of the negations appear.

• Distributions of identical objects into distinct bins: For a given n is easy to count the number of solutions to $a+b+c=n$ with given lower bounds on a,b,c. But it is harder to count the number of solutions to $a+b+c=n$ with given lower bounds and upper bounds on a,b,c. (Satisfying an upper bound x is the same as not satisfying a lower bound x.)

• Hands with forbidden denominations: It is easy to count the number of hands that do not have a particular denomination, or denominations— just deal from a smaller deck! But it is harder to count the number of hand that do have a particular denomination, or have, say, exactly three out of eight specific denominations in a hand of six cards.

P.I.E. to the rescue!

(Draw diagrams!) Let U be a universal set (say arrangements of letters, or distributions of objects), and let $A, B, C, \ldots$ be subsets of U. We can also think of $A, B, C, \ldots$ as conditions that elements of U may or may not satisfy — the set A consists of exactly the elements satisfying condition A.

Let N_U be the number Let N_A be the number of objects in set A (and possibly other sets), the number of objects satisfying condition A (satisfying *at least* condition A). Similarly let N_B the number of objects at least in B, etc, etc.

Next let N_{AB} be the number of objects in set $A \cap B$, the number satisfying condition A *and* satisfying condition B (satisfying *at least* those conditions), and the same for all pairs.

And let N_{ABC} be the number of objects in set $A \cap B \cap C$, or satisfying condition $A \wedge B \wedge C$, and of course the same for any $AB \ldots K$.

Notice that if $\cap$ for sets and $\wedge$ for conditions are treated as multiplication, it is natural to refer to ABC, $ABCD$, or more generically $AB \ldots K$.

The Principle of Inclusion-Exclusion is useful to us when we can easily calculate N_*'s where $*$ is any $AB \ldots K$ (even just A, etc.), but we are interested in how many elements have *none* of these condition. PIE will do more, too, and for each n, help us count how many elements satisfy exactly n condition.

The Principle of Inclusion-Exclusion (version 1)

For subsets or condition $A, B, \ldots, K$, the number E_0 of elements satisfying exactly *zero* conditions is

$$E_0 := N_U \;-\; (N_A + N_B + \ldots) \;+\; (N_{AB} + N_{AC} + \ldots) \;-\; (N_{ABC} + \ldots) \;+\; \ldots$$

1. a) Draw a Venn diagram with universal set U and subsets A and B.
   b) Suppose you know the sizes N_U, N_A, N_B and N_{AB}. Verify the correct value of E_0, the number elements not in $A \cup B$, satisfies P.I.E.
   c) Among twenty-four sophomore eight are taking Biology and nine are taking Chemistry. Two are taking both. How many are taking neither? How many are taking just one or the other?
   d) Among nineteen pea plants, thirteen express gene A and nine express gene B. What is the maximum possible number of plants expressing both? The minimum? The max and min of those expressing neither?
2. Draw a Venn diagram with universal set U and subsets A, B and C.

3. Suppose you know the sizes N_U, N_A, N_B, N_C, N_{AB}, N_{AC}, N_{BC}, and N_{ABC}. What is

   a) E_0, the number of elements not in any of A, B, C?

   b) Among twenty four sophomores, ten are taking (A)lgebra, eight are taking (B)iology and nine are taking (C)hemistry. In fact:

set S	N_S
A	10
B	8
C	9
AB	4
AC	3
BC	2
ABC	1

   Fill in a Venn diagram with numbers.

   c) How many are taking none of Algebra, Biology nor Chemistry? 5

   d) How many are taking just one class. Write this in terms of the $N_*'s$. 12

   e) How many are taking exactly two classes? Write this in terms of the $N_*'s$. 6

   f) Try this the other way round. Draw a Venn diagram with three overlapping sets A, B, C. Place numbers in each of the 2^3 regions of the diagram. What are N_A, N_{AB} etc, and N_0, N_1, etc, and E_0, E_1, etc?

.

Recall that for any $n \geq 0$,

$$E_0 := N_U \;-\; (N_A + N_B + \ldots) \;+\; (N_{AB} + N_{AC} + \ldots) \;-\; (N_{ABC} + \ldots) \;+\; \ldots$$

Let us give names to these terms: let

$N_0 := N_U$, the number of objects in the universal set U, or in other words, the number of objects satisfying at least none of the conditions.

$N_1 := N_A + N_B + \ldots$, the sum over all the conditions or subsets.

$N_2 := N_{AB} + \ldots$, the sum over all possible pairs of conditions or subsets.

N_3, the sum over all triples,

etc.

Then

$$E_0 = +N_0 - N_1 + N_2 - N_3 \dots$$

More generally (and for a direct proof see[14]):

$$\begin{aligned}
E_1 &= \ + \quad N_1 - \tbinom{2}{1} N_2 + \tbinom{3}{1} N_3 - \tbinom{4}{1} N_4 + \dots \\
E_2 &= \ +\tbinom{2}{2} N_2 - \tbinom{3}{2} N_3 + \tbinom{4}{2} N_4 - \tbinom{5}{2} N_5 + \dots \\
E_3 &= \ +\tbinom{3}{3} N_3 - \tbinom{4}{3} N_4 + \dots \\
\dots \\
E_n &= \ \sum_{i=0}^{\infty} (-1)^i \tbinom{n+i}{n} N_{n+i}
\end{aligned}$$

4. Check these formulas for the examples above, the number of elements with exactly one of A, B, C, or exactly two, in terms of N_A, N_{AB} etc.
5. In a pool of 80 medical study patients, 15 have condition `A` (and possibly other conditions, as for all of these), 29 have at least condition `B`, 34 have condition `C` and 38 have condition

[14] Soon the use of generating functions will give us a very nice proof. Here is a direct count:

Let $n \geq 1$, and consider the sum $\sum_{i=0}^{\infty} (-1)^i \binom{n+i}{n} N_{n+i}$— in particular, how much does each object in U contribute?

Suppose that an object has k properties. If $k < n$, then that object is not counted in any of the terms N_{n+i}, and contributes nothing. If $k = n$, then that object is counted exactly once in N_n, and not at all in the remaining terms, and so is counted once overall.

Suppose $k > n$. Then for each i, the object is included in N_{n+i} in $\binom{k}{n+i}$ different ways — the object has k properties and N_{n+i} is the sum of each N_* for collections $*$ of $(n+i)$ properties.

Consequently an object with $k > n$ properties is counted

$$\begin{aligned}
\sum_{i=0} (-1)^i \binom{n+i}{n} \binom{k}{n+1} &= \sum (-1)^i \frac{(n+i)!}{n!i!} \, \frac{k!}{(n+i)!(k-n-i)!} = \frac{k!}{n!}(k-n)! \sum (-1)^i \frac{(k-n)!}{(k-n-i)!i!} \\
&= \binom{k}{n} \ (1 + (-1))^{(k-n)} = 0
\end{aligned}$$

times!

Thus, totaling across all objects, those objects with exactly n properties are counted once each, and the others contribute nothing. The sum is E_n.

D. Moreover 7 have A & B, 7 have A & C, 5 have AD, 18 BC, 12 BD and 17 CD. Furthermore, 2 patients have conditions A, B & C, 3 have ABD, 3 have ACD and 8 have BCD. Finally 1 patient has all four conditions, ABCD.

a) How many patients have *none* of the conditions? 15

b) How many patients have exactly one of the conditions? 28

c) How many patients have exactly two of the conditions? 24

d) How many patients have exactly three of the conditions? Four? 12,1

$\sim 4.033\ 10^{26}$

6. How many numbers from 1 to 1000 are not divisible by any of $2, 3$ and 5? 266

7. Twelve jelly donuts are distributed to four children, Alice, Bob, Cynthia and Daniel. If each child has at least one, but no more than four donuts, how many ways can the donuts be distributed?

   Let's spell this one out, and then give a few similar problems. There are four conditions: that Alice has too many donuts, that Bob has too many, that Cynthia does, that Daniel does.

   a) It is easy to count N_A, N_B, etc — do so! What is N_1? 140

   b) Count N_{AB}, N_{BC}, etc. How many of these terms are there? What is N_2? 6

   c) What is N_3? 0

   d) What is N_4, and don't forget N_0. 0,165

   e) Now how many ways can *exactly none* of these conditions be satisfied, no child has too many donuts? 31

   This will be the same process for the next few:

8. Twenty identical screwdrivers are to be distributed to five different work stations. How many ways may this be done if each work station may have 3, 4, or 5 screwdrivers? •

9. Thirty balls drop into five distinct buckets. What is the probability that no bucket has more than ten balls? $\sim 15.20\%$

10. A solution of non-negative integers to $a + \cdots + z = 26$ is chosen at random. What is the probability that all the variables are no bigger than 5? $\sim 68.72\%$

    The next few problems restricting the number of suits or denominations in die rolls or hands of cards:

11. Ten cards are chosen from a standard deck. How many ways can this be done in such a way that

    a) There is at least one card from each suit? 13308911902

    b) At least one suit is not chosen? 2511112318

    c) Exactly one suit is not chosen? 2479244196

12. Six cards are chosen from a standard deck. How many ways can this be done so that there are exactly four denominations? What is the probability that there are exactly four denominations?

13. Seven six-sided dice are rolled. What is the number of ways that exactly four different values may appear on the dice? 126000

14. Six twenty-sided dice are rolled. How many ways can this be done so that there are exactly four values showing? What is the probability that there are exactly four values showing?

15. How many strings can be formed from the thirteen letters `COMBINATORICS`

    a) How many strings so that (at least) the substring `OO` appears? So that `CC` appears? `II`?

    b) What is N_1? 359251200

    c) How many strings in which *both* `OO` and `CC` appear?

    d) How many pairs are there? What is N_2? 59875200

    e) How many strings so that the substrings `OO` `CC` and `II` appear?

    f) What is N_3? N_4? N_0? 3628800 0 778377600

    g) Using *P.I.E.*, what is E_0, the number of strings in which *none* of the substrings `OO`, `CC` or `II` appear? 475372800

    h) What is E_1, the number of strings in which *exactly one* of the substrings `OO`, `CC` or `II` appears? 250387200

    i) What is E_2, the number of strings in which *exactly two* of the substrings `OO`, `CC` or `II` appears? 48988800

    j) What is E_3? Does this fit the formula for P.I.E.? (What are $N_4, N_5, \ldots$?) 3628800,0

16. Consider the 26! arrangements of the letters `A` to `Z`.

    a) How many have all of the substrings `ABC`, `DEF` and `GHI`? $\sim 2.433\ 10^{18}$

    b) How many have at least both of `ABC`, `DEF` (and possibly or not `GHI`) ? Similarly `DEF`, `GHI` ? etc. $\sim 1.124\ 10^{21}$

    c) How many have at least `ABC` (and possibly or not the others)? Etc? $\sim 6.204\ 10^{23}$

    d) How many have none of the substrings `ABC`, `DEF` and `GHI`? $\sim 4.014\ 10^{26}$, or 99.5%

    e) How many have exactly one of the substrings `ABC`, `DEF` and `GHI`?

17. An arrangement of the fifty-two letters `A A` through `Z Z` is chosen at random. Do you bet there is, or do you bet there isn't a pair of identical of identical adjacent letters? $\sim 36.43\%$

PIE and derangments

A *permutation* of a set is just a bijection from the set to itself — in other words, every thing in the set is sent somewhere. In a permutation, it's perfectly possible that something is sent to itself (is a "fixed point"), and a basic question is, if a permutation is chosen at random, what is the probability it has no fixed points, is a "derangement"? Let us answer this:

18. By hand, list the permutations (arrangements) of the digits `1`, `2`, and `3`, and identify how many have no fixed points (`1` is not in the `1`st position, etc.) By listing them out, count how many of the 24 permutations of `1`, `2`, `3`, `4` are derangements.
19. Nine students put their ID cards into a hat, and then each chooses an ID at random.[15]
    a) What is the probability that no student has their own ID?

       For this we need to work out some sub-problems. Suppose the students' names are `Alice` through `Ira`. The nine conditions are that `Alice` gets her own ID, that `Bob` gets his own ID, . . . that `Ira` gets their own ID.

       a) What is N_0, or in other words, what is the number of ways that at least none of the conditions are satisfied— what is the total number of ways the students can draw the IDs from a hat? 362880
       b) What is the number of ways that (at least) the first condition can be satisfied, that `Alice` draws her own ID, $N_{\mathtt{A}}$ 40320
       c) Similarly for each student `X`, $N_{\mathtt{X}}$ is the same. There are nine students, $\binom{9}{1}$, and so give $N_1 = N_{\mathtt{A}} + \ldots N_{\mathtt{I}}$.
       d) What is $N_{\mathtt{A}B}$, the number of ways `Alice` and `Bob` both draw their own ID's (and others may or may not)?
       e) Similarly for all pairs `XY`, N_{XY} is the same. Multiply by the number of pairs and give N_2.
       f) Give N_3, and for each $n \leq 9$, give N_n.
       g) Using PIE, give an expression for E_0. Cancel out what you can but don't oversimplify the arithmetic, so that you can:
       h) Show that E_0/N_0 is very nearly $1/e$ (!!)

    b) What is the probability that exactly one student has their own ID (think about it, and check using PIE).
    c) two?
    d) any $n \leq 9$?
20. Let us work out the general formulas. Consider permutations on the n objects `1`, `2`, . . . `n`. There are n conditions. What are:
    a) N_0

[15] This strange premise is stereotypical for problems on derangements.

b) N_1

c) N_2, etc

d) E_0, the number of derangements

e) E_0/N_0, especially as the number n of objects tends towards infinity.

f) Similarly, what is $\lim_{n\to\infty} E_k/N_k$?

21. A simple cipher is just a derangement of the letters of the alphabet, each letter being replaced with some other letter.

    a) To a close approximation, how many simple ciphers are possible in the twenty-six letter roman alphabet `A`, ... `Z`?

    b) How many even simpler ciphers are possible if each of the five vowels must be replaced with a vowel and each of the 21 consonants must be replaced with a consonant?

"Sya veka xova o dumyak up, sya aopuak us up se nadumyak us!"

5 Generating functions

A **sequence** $a_0, a_1, ...$ is simply a list of numbers, which we'll abbreviate (a_k). [16] Generating functions are a good way to manipulate and work with sequences.

There are a few different kinds of generating functions, but we will mostly work with "ordinary" generating functions, and touch on with "exponential" gf's. (There are other kinds of generating functions as well.)

A function $A(x)$ is the **ordinary generating function** for a sequence (a_k) iff $A(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots$.

We'll write $(a_k) \overset{\text{ogf}}{\longleftrightarrow} A(x)$ to mean that $A(x)$ is the ordinary generating of the sequence (a_k).

In other words, $(a_k) \overset{\text{ogf}}{\longleftrightarrow} A(x)$ if and only if each $a_k = [x^k]A(x)$, the coefficient of x^k in the expansion of $A(x)$.

An expression is in **closed form** iff it is expressed using a finite number of standard operations and no reference to infinity. We will try to express our generating functions in closed form.

Generating functions are a kind of way of encoding sequences of numbers, as Taylor series, and manipulating the results to work out any number of combinatorial problems, in surprisingly sneaky ways!

For example, the number of ways to split a number \$$n$ into \$5 and \$10 bills is exactly the coefficient of x^n in the expansion of

$$((x^5)^0 + (x^5)^1 + (x^5)^2 + (x^5)^3 + \ldots) \cdot ((x^{10})^0 + (x^{10})^1 + (x^{10})^2 + (x^{10})^3 + \ldots)$$
$$= 1 + x^5 + 2x^{10} + 2x^{15} + 3x^{20} + 3x^{25} + 4x^{30} + \ldots$$

There are four[17] ways to split \$30, as \$10,10,10, or as \$10,10,5,5, or as \$10,5,5,5,5, or as all \$5's.

How does this magic work? Because the coefficient 4 of x^{30} records the four ways to choose a term from $((x^5)^0 + (x^5)^1 + (x^5)^2 + (x^5)^3 + \ldots)$ and a term from $((x^{10})^0 + (x^{10})^1 + (x^{10})^2 + (x^{10})^3 + \ldots)$ to multiply to get x^{30}.

[16] We will almost always index our sequences starting at 0 — so be a little careful!

[17] oh yeah, there are no ways to split \$17 into \$5's and \$10's and the coefficient of x^{17} is, uh, 0

Since those are geometric series, we can write this much more succinctly: Remember that for $|x| < 1$, $1 + x + x^2 + x^3 + \ldots = 1/(1-x)$. For arbitrary values of x, this won't make literal sense, but we *formally* declare

$$1 + x + x^2 + x^3 + \ldots :=: \frac{1}{1-x}$$

and so $((x^5)^0 + (x^5)^1 + (x^5)^2 + (x^5)^3 + \ldots) = 1/(1-x^5)$.

The number of ways to split \$$n$ into \$5 and \$10's is the coefficient of x^n in the expansion of

$$\frac{1}{(1-x^5)(1-x^{10})}$$

which is easier to work with and leverage.

5.1 Exercises

1. a) Let $(a_k) = \binom{4}{k}$, the sequence $1, 4, 6, 4, 1, 0, 0, \ldots$.. What is the (closed form) generating function for this sequence?

   b) More generally, for an arbitrary counting number n, what is the closed form generating function for the sequence (a_k) with $a_k = \binom{n}{k}$.

2. For some values x, we can literally write the infinite sum $1 + x + x^2 + \ldots$ as $1/(1-x)$, a simpler expression. Though this doesn't exactly make sense for all x, we simply pretend that it does, and *define* $1 + x + x^2 + \ldots = 1/(1-x)$. Write the following series in simpler finite, "closed" form:

   a) $1 + x^2 + x^4 + x^6 + \ldots$

   b) $x + x^3 + x^5 + x^7 + \ldots$

   c) $x^3 + x^4 + x^5 + x^6 + \ldots$

   d) $1 + (x/4) + x^2/16 + x^3/64 + \ldots$

   e) $(1+x+x^2+x^3+\ldots)\cdot(1+x+x^2+x^3+\ldots)$

   f) $(1 + x^3 + x^6 + \ldots) \cdot (x^2 - x^3 + x^4 - x^5 \ldots)$

3. What are (closed form) ordinary generating functions for the following sequences:

   a) 1,1,1,1,1,....

   b) 1,2,4,8,16,...

   c) 1,-1,1,-1,1,...

   d) 0,0,1,1,1,1,....

   e) 1,0,1,0,1,0,....

   f) 0,1,0,1,0,1,....

g) 0,1,0,0,1,0,0,1,...

h) 2,6,18,54,162,...

i) 7,-14,28,-56,112,...

j) $a_k = 3^k - 2 \cdot 4^k$

4. What sequences are generated by:

a) $1/(1-3x)$

b) $1/(2+x)$

c) $(1+3x)^5$

d) $1/(1+3x)^5$

e) $(x-1)/(1-3x)$

f) $1/(1-x^2)$

g) $1/(1+x)(1-x)$

h) $1/(1-2x-3x^2)$

i) $1/(1-2x+x^2)$

For the last three of these, you will need to use the method of partial fractions.

5. A helpful identity. You already know this from examples. Explain why

$$[x^n]x^k f(x) = [x^{n-k}]f(x)$$

and

$$[x^{n+k}]f(x) = [x^n]\frac{f(x)}{x^k}$$

For example, what is the coefficient of x^n in $x^2/(1-2x)^3$?

6. Let's look again at distributing n identical objects into k distinct categories, the number of solutions to $a + \cdots + k = n$. Generating functions will give us another way to count these solutions.

a) For a fixed counting number n, the number of solutions are there to

$$a + b + c + d = n, \qquad \text{with } a, b, c, d \geq 0$$

is *exactly*

$$[x^n](1 + x + x^2 + ...)^4 = [x^n]\frac{1}{(1-x)^4}$$

Explain. Check that this agrees with our previous count.

b) Let's change some of the conditions: write a generating function for the number of solutions, for each given n, of

$$a + b + c + d = n, \qquad \text{with } a \geq 0, b \geq 5, c \geq -2, d \geq 3$$

From the generating function calculate the number of solutions for each n.

c) Write a generating function for the number of solutions, for each given n, of

$$a + b + c + d = n, \qquad \text{with } 0 \leq a, b, c, d \leq 4$$

(The generating function is not hard, but it's less friendly to obtain its coefficients; use "Expand" in Wolfram alpha.

7. (a) How many ways are there to distribute n identical donuts to Alice, Bob, Cindy and Daniel, if each child should get at least one donut?

   (b) How many ways are there to distribute n identical donuts to Alice, Bob, Cindy and Daniel, if each child should get at least one donut but no more than three?

8. Ms. Witham will distribute n pennies to Alice, Bill, Carl, Dora and Edward; Alice and Bill (her favorites) will get at least five pennies, and all the others will get at least one penny. But Edward (the rascal!) will get no more than three.

   - How many ways may she distribute $n = 20$ pennies?
   - Write, in closed form, an ordinary generating function for the number of ways that n pennies may be distributed.

9. Here's a batch of problems that are, in fact, all the same. Give closed form generating functions for:

   a) the number of ways there are to break \$$n$ into change using \$1, \$3 and \$7 bills?

   b) the number of solutions there are to $a+b+c = n$ with each $a, b, c \geq 0$, and b is a multiple of 3 and c is a multiple of 7?

   c) the number of solutions there are to $x + 3y + 7z = n$ with each $x, y, z \geq 0$?

   d) Using Wolfram Alpha or some other symbolic calculator, find the number of solutions with $n = 25$. 22

10. The **summation operator** is very useful. Prove (or at least justify) that if $f(x)$ is the ordinary generating function for (a_k), then $f(x)/(1 - x)$ is the ogf for the sequence of sums a_0, $(a_0 + a_1)$, $(a_0 + a_1 + a_2)$, $\ldots$.

    In other words, show that

$$[x^k]f(x)/(1-x) = \sum_{0}^{k} a_i$$

    And try an example.

11. Using the summation operator, find the ordinary generating function for the sequences

    a) $1, 2, 3, 4, 5...$

    b) $0, 1, 4, 9, 16, ...$ (What sequence is this the sum of?)

    c) $0, 1, 8, 27, 64, ...$

d) On the nth day of Christmas[18], my true love gave to me n X_n's, $(n-1)$ $X_{(n-1)}$'s $\ldots$ two X_2's and one X_1. (On the 0th day of Christmas, of course, my true love gave me nothing.)

   i) Give the ordinary generating function for the number of things my true love gave to me on day n.
   ii) Give the ordinary generating function for the total number of things my true love has given me up through day n.
   iii) What is the exact total number of gifts given up through day n? How fast does this total grow as n grows?

12. Let's show that $1 + 2 + 2^2 + \ldots 2^n = 2^{n+1} - 1$ using the summation operation.

   a) Give a ogf for the sequence $1, 2, ..., 2^n, ...$
   b) Using the summation operator, give an ogf for the sequence $1, (1 + 2), (1 + 2 + 4), ...(1 + 2 + 2^2 + \ldots 2^n), ...$
   c) Using partial fractions, split apart this ogf and show it also generates $(2^1 - 1), (2^2 - 1), ..., (2^{n+1} - 1), \ldots$

13. Generating functions easy to find for the number of partitions of a number (though it is usually difficult to get a formula for the exact coefficients; use Wolfram Alpha to expand some of these). How many partitions of the counting number n are there if *(wait: try out some examples of these with small n— can you count the partitions out?)*

   a) The summands are just counting numbers;
   We'll give you this one:

$$(1+x+x^2+...)(1+x^2+x^4+...)(1+x^3+x^6+...)... = \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}.... \qquad (*)$$

   or more precisely

$$\prod_{n=1}^{\infty}\sum_{k=0}^{\infty} x^{nk} = \prod_{n=0}^{\infty}\frac{1}{1-x^n}$$

   For example, the eleven partitions of 6 are

   - 6
   - 5 1
   - 4 2
   - 4 1 1
   - 3 3
   - 3 2 1
   - 3 1 1 1
   - 2 2 2
   - 2 2 1 1
   - 2 1 1 1 1
   - 111111

[18]On the twelfth day of Christmas my true love gave to me: twelve drummers drumming, eleven pipers piping, ten lords a'leaping, nine ladies dancing, eight maids a'milking, seven swans a'swimming, six geese a'laying, five golden rings, four calling birds, three French hens, two turtle doves, and a partridge in a pear tree. Then on the *thirteenth* day of Christmas...

In the expansion of the above, we have

$$\begin{aligned}(*) = \quad & ... + (x^6)^1 \cdots + (x^5)^1(x^1)^1 \cdots + (x^4)^1(x^2)^1 \cdots + (x^3)^2 \cdots + (x^3)^1(x^2)^1(x^1)^1 \ldots \\ & \quad +(x^3)^1(x^1)^3 \cdots + (x^2)^3 \cdots + (x^2)^2(x^1)^2 \cdots + (x^2)^1(x^1)^4 \cdots + (x^1)^6 \\ & = \ldots + 11x^6 + \ldots\end{aligned}$$

The algebra of the expansion captures the logic of the possible partitions. If we want to restrict the possibilities for our partitions, we just strike out the corresponding terms. Here are more examples:

b) the summands cannot be greater than 5;

c) the summands are odd counting numbers;

d) the summands are distinct;

e) the summands are distinct, and from 1 to 9.

14. Euler showed the number of partitions of n into *odd* summands is the same as the number of partitions of n into *distinct* summands, by showing the generating functions in **??** (c) and (d) are algebraically the same. Do the same. (Hint: multiply each term of the generating function (d) by $(1-x^k)/(1-x^k)$; apply the difference of squares, and simplify. This is a super cool proof.

15. Here's another way to get generating functions for sequences like n, or n^2 or $3n^3 + n - 7$.

    a) The function $f = 1/(1-x)$ generates the sequence $1, 1, 1, \ldots..$ Show that $\frac{d}{dx}f$ generates $0, 1, 2, 3, ...$ and that $x\frac{d}{dx}f$ generates $1, 2, 3, 4, 5...$ We can write this as $[x^n](x\frac{d}{dx})f = n$.

    b) Show that $(x\frac{d}{dx})^2 f$ generates the sequence n^2 (where $(x\frac{d}{dx})^2 f$ means $x\ \frac{d}{dx}(x\frac{d}{dx}f)$). Check this matches your answers to b) above.

    c) In general, if $P(x)$ is a polynomial, show the nice fact that $P(x\frac{d}{dx})f$ generates the sequence $P(n)$.

    d) Explicitly give closed form generating functions for the sequences

        i) $(n^2 + n)/2$ (the triangular numbers)

        ii) $n(n+1)(2n+1)/6 = (n + 3n^2 + 2n^3)/6$ (the tetrahedral numbers)

        iii) n^3

5.2 Generating Functions and PIE

Consider a set, and subsets defined by some conditions. For each collection $\mathcal{C}$ of k of these conditions, it is easy to count the number $N_{\mathcal{C}}$ of ways that at least those conditions are satisfied. Let N_k be the sum of these numbers,

$$N_k := \sum_{\substack{\text{collections } \mathcal{C} \\ \text{of } k \text{ conditions}}} N_{\mathcal{C}}$$

and let $\mathcal{N}(x)$ be the generating function

$$\mathcal{N}(x) = \sum N_k x^k$$

Let E_k be the number of ways that *exactly* k conditions are satisfied and let $\mathcal{E}(x)$ be the generating function for the E_k's, in other words that

$$\mathcal{E}(x) := E_0 + E_1 x + E_2 x^2 + E_3 x^3 + ...$$

The Principle of Inclusion-Exclusion is useful when we are able to work out the N_k's but we really want to know the E_k's. P.I.E. shows the way!

To get started, lets try to write each N_k in terms of the E_k's. (That gives us a system of equations, that we'll then try to solve.)

Suppose an object satisfies exactly k conditions. How many times would it be counted in N_j, where $j \leq k$? N_j is summed up over all the choices of j conditions; an object that satisfies k conditions will be counted $\binom{k}{j}$ times. This is so for every object counted in E_k. Thus

$$N_j = \sum_k \binom{k}{j} E_k$$

Now let's look at the generating functions $\mathcal{N}(x)$ and $\mathcal{E}(x)$ (dropping all the $\binom{k}{l}$'s equal to 0).

$$\begin{aligned}
\mathcal{N}(x) &= N_0 x^0 + N_1 x^1 + N_2 x^2 + ... \\
&= x^0 \left(\tbinom{0}{0} E_0 + \tbinom{1}{0} E_1 + \tbinom{2}{0} E_2 + \dots\right) \\
&\quad +x^1 \left(\tbinom{1}{1} E_1 + \tbinom{2}{1} E_2 + \tbinom{3}{1} E_3 + \dots\right) \\
&\quad +x^2 \left(\tbinom{2}{2} E_2 + \tbinom{3}{2} E_3 + \tbinom{4}{2} E_4 + \dots\right) \\
&\quad \dots
\end{aligned}$$

All we have to do is solve for the E's in terms of the N's! But, gathering up the E_i's, we can rearrange this as

$$\begin{aligned}\mathcal{N}(x) &= (\tbinom{0}{0}x^0)E_0 \\ &+(\tbinom{1}{0}x^0 + \tbinom{1}{1}x^1)E_1 \\ &+(\tbinom{2}{0}x^0 + \tbinom{2}{1}x^1 + \tbinom{2}{2}x^2)E_2 \\ &+(\tbinom{3}{0}x^0 + \tbinom{3}{1}x^1 + \tbinom{3}{2}x^2 + \tbinom{3}{3}x^3)E_3 \\ &+\dots\end{aligned}$$

or more succinctly:

$$\mathcal{N}(x) = (1+x)^0E_0 + (1+x)^1E_1 + (1+x)^2E_2 + \dots (1+x)^nE_n + \dots$$

In short, $\mathcal{N}(x) = \mathcal{E}(1+x)$. Consequently,

$$\mathcal{E}(x) = \mathcal{N}(x-1)$$

Let's follow some conclusions of this, explicitly find the formulas, and prove PIE holds:

Suppose we know $N_0, N_1, \dots.$, and

$$\mathcal{E}(x) = \mathcal{N}(x-1) = N_0(x-1)^0 + N_1(x-1)^1 + \dots + N_k(x-1)^k + \dots$$

Then

16. Expand out each $N_n(x-1)^n = N_n \sum(-1)^k\binom{n}{k}x^k$. In the sum $\mathcal{E}(x) = \mathcal{N}(x-1)$, gather up the terms E_nx^n— the coefficients are just what we have been seeking. Each E_n is expressible as the summation of terms $\pm\binom{k}{n}N_k$:

    a) $E_0 =$
    b) $E_1 =$
    c) $E_2 =$
    d) $E_3 =$

e) $E_k =$

This[19] gives a nice proof of PIE, but in practice its application remains as before.

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Thus

$$\begin{aligned}
\mathcal{N}(x-1) &= \sum (x-1)^k N_k \\
&= (x-1)^0 N_0 + (x-1)^2 N_1 + ... \\
&= x^0 \ \sum_i \binom{i}{0}(-1)^{i-0} N_i \\
&\quad +x^1 \ \sum_i \binom{i}{1}(-1)^{i-1} N_i \\
&\quad +x^2 \ \sum_i \binom{i}{2}(-1)^{i-2} N_i \\
&\quad + \dots \\
&\quad +x^k \ \sum_i \binom{i}{k}(-1)^{i-k} N_i \\
&\quad + \dots
\end{aligned}$$

Therefore, for any k, $E_k = \sum_i \binom{i}{k}(-1)^{i-k} N_i$.

5.3 Rook Polynomials, an application of generating functions

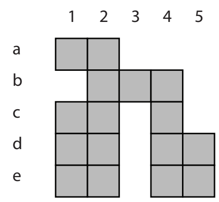
How many ways can a bunch of complicated conditions be satisfied?

An example

Suppose we are pairing up items in $\{A, B, C, D, E\}$ to items in $\{1, 2, 3, 4, 5\}$, but subject to some weird complicated conditions. To represent this diagramatically:

We can only match		
A	to	1 or 2
B	to	$2, 3$ or 4
C	to	$1, 2,$ or 4
D	to	$1, 2, 4,$ or 5
E	to	$1, 2, 4,$ or 5

we might draw



These are perfectly reasonable questions:

- Can we match each of A, B, C, D and E with $1, 2, 3, 4,$ and 5, satisfying these conditions? In other words, can we fit 5 rooks into this board?
- If we can, in how many ways may we do this?
- Or to change it up: How many ways may we fit 3 rooks in, with none on the same row or column as any other?
- How can we systemize this?

Another Example

Three inputs XYZ must link to one of four outputs $STUV$. No output may be linked to from more than one input. X may link to STU, Y to TUV, and Z to SV. How many ways may two of the three inputs be linked?

1. Draw a board representing this problem.

Another Example

Six teams A, B, C, D, E, F are available to be paired with four clients, W, X, Y, Z (Two teams will not be needed.) Teams A, B, C are experts on the needs of W, X, Y. Teams D, E work well with Y, Z. Team F will only work with team W. How many ways may the four clients be matched up with a team?

2. Draw a board representing this problem.

The general picture

Without getting into any formalisms, we want to count how many ways *rooks* may be placed on a *board*, no two rooks occupying the same row or column.

Another way to put this is that we are counting the number of ways to choose matchings between a pair of sets, subject to some conditions. The sets are the horizontal rows and the vertical columns. To choose a rook is to match up a row and a column. The conditions describe which matchings are permitted and which are forbidden — which squares are in the board and which are not.

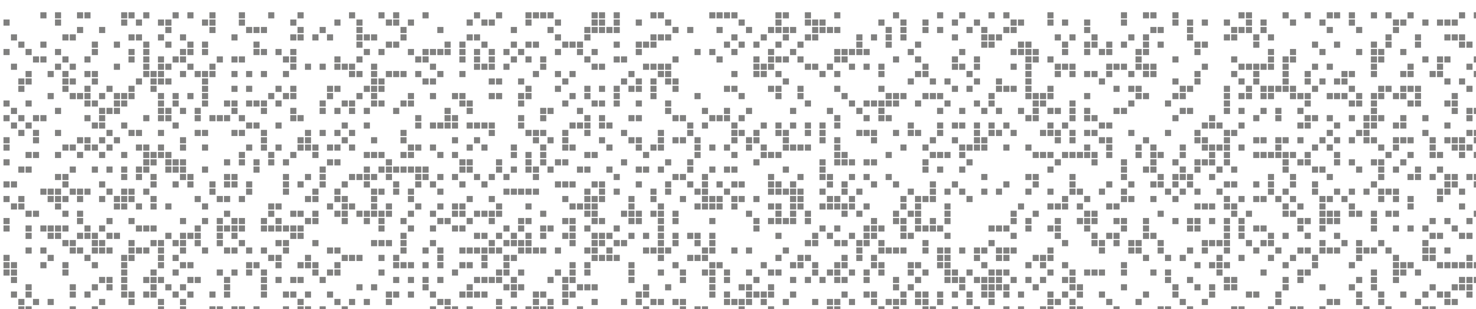
Fixing a board, for each $k \geq 0$, we let r_k be the number of ways to place k rooks. You can guess that $r_0 = 1$ on every board.[20]

3. Things will be more complicated quickly. By hand, calculate the sequence $r_1, r_2, r_3, r_4, \ldots$ for the boards in the examples above.

Do you get these numbers?

	r_0	r_1	r_2	r_3	r_4	r_5
ex. 1	1	16	78	135	74	8
ex. 2	1	8	17	8		
ex.3	1	14	60	92	42	

4. Here is a very complicated 40×250 board, with 2473 available squares. At least we can easily see: what is r_1, the number of ways to place one rook on this board?



For any given board B, we consider the generating function

$$P(B) := r_0 + r_1 x + r_2 x^2 + \ldots$$

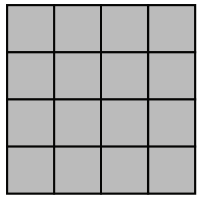
of this sequence r_0, $r_1, r_2 \ldots, r_k \ldots$. These generating functions $P(B)$ are called *rook polynomials*.

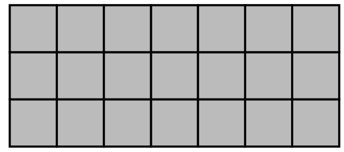
5. Explain why these *are* polynomials, that there are only finitely many non-zero coefficients r_k on a finite board B

The rook polynomial of a complete board

[20] As you'll see in a moment, this is the sensible value for even an empty board.

We'll call an $m \times n$ rectangle of unit squares the "complete $m \times n$ board", and denote this $B_{m\times n}$. Here are $B_{4\times 4}$ and $B_{3\times 7}$:





6. In an $n \times n$ board $B_{n\times n}$, how many ways can
   (a) one rook be placed?
   (b) n rooks be placed?
   (c) two rooks be placed? (Choose the columns, then the rows...)
   (d) k rooks be placed, for each $1 \le k \le n$?
   (e) Give the rook polynomial $P(B_{n\times n})$
7. On an $m \times n$ board $B_{m\times n}$,
   (a) how many ways can k rooks be placed?
   (b) Does the formula work out correctly to 0 if k is too large?
   (c) Give the rook polynomial $P(B_{m\times n})$.

A recursive way to calculate rook polynomials

There's an easy algorithm to calculate rook polynomials! (Easy to understand that is — but intractably lengthy for large boards.)

We are interested in the ways to place k rooks on a board; lets call these ways *placements*.

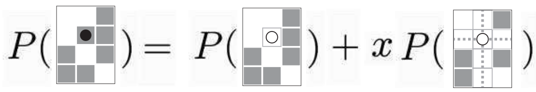
The insight is that the placements of k rooks on a board B can be split into two cases, corresponding to placements of rooks on smaller boards. These placements in turn split into smaller cases still, and so on, until we have reduced the problem to counting on the very simplest boards of all.

Fix a board B, like the one at left below, and choose any square within it. (Any square will work for the algorithm but a human who wants to save time should be strategic!) Let B° be the board with the special square deleted. Let $B^\dagger$ be the board with the entire row and column of the special square deleted. Then

Theorem:

$$P(B) = P(B^\circ) + xP(B^\dagger)$$

For example,



Proof: It suffices to prove that the coefficients of the polynomials on the right and left are the same. We need to show that

$$\begin{aligned}[x^n]P(B) &= [x^n](P(B^\circ) + xP(B^\dagger)) \\ &= [x^n]P(B^\circ) + [x^{n-1}]P(B^\dagger)\end{aligned}$$

But this is clear: $[x^n]P(B)$ is the number of ways to place n rooks on the board B. These ways split into two cases: there is not a rook on the special square or there is a rook on the special square.

The number of ways in the first case is exactly the number of ways to place n rooks on $P(B^\circ)$, in other words, exactly $[x^n]P(B^\circ)$.

In the second case, a rook has already been placed on the special square, and its row and column are no longer available for the remaining $(n-1)$ rooks, which may be placed on $B^\dagger$, in $[x^{n-1}]P(B^\dagger)$ ways. The coefficients of the polynomials are thus equal, and so too are they: $P(B) = P(B^\circ) + P(B^\dagger)$.

It's also helpful to notice that:

Theorem Let B be the disjoint union of two subboards, B_1 and B_2, that share no column and share no row. Then $P(B) = P(B_1)P(B_2)$.

Proof: This is standard issue generating function stuff! The set of all placements of n on B can be partitioned into many subcases: no rooks on B_1 and n rooks on B_2; one rook on B_1 and $(n-1)$ rooks on B_2, ..., ...and finally n rooks on B_1 and no rooks on B_2. But this is perfectly captured as the product of the two rook polynomials.

Taking r_n to be the number of ways to place n rooks on B, and s_k, t_k to be the number of ways to place k rooks on B_1, B_2 respectively, we have

$$r_n = s_0t_n + s_1t_{n-1} + \cdots + s_nt_0 = \sum_{i+j=n} s_it_j$$

and so

$$(r_0 + r_1x + r_2x^2 + \ldots) = (s_0 + s_1x + s_2x^2 + \ldots)(t_0 + t_1x + t_2x^2 + \ldots)$$

Thus

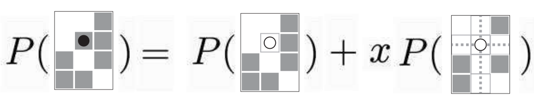
$$P(B) = P(B_1)P(B_2)$$

Finally

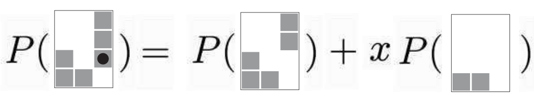
Theorem: Let B and B' be boards that differ only by permuting their rows and permuting their columns. Then $P(B) = P(B')$.

8. In order for the recursion to be correct, what must be the rook polynomial of the empty board, $R(\emptyset)$? (Hint: Consider the recursion starting from $B_{1\times 1}$.)

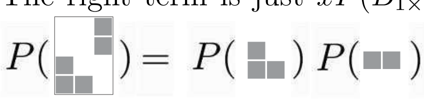
Let's work through an entire example. To calculate the rook polynomial of the board at left, we choose (arbitrarily) a special point. We have



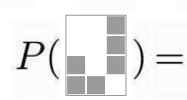
To calculate with these, we break the boards up further. We have



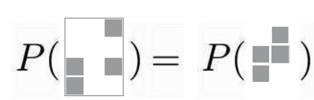
The right term is just $xP(B_{1\times 2}) = x(1 + 2x)$. The left term is a product of two disjoint boards:



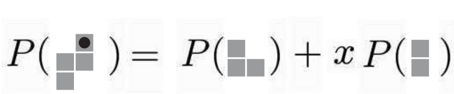
One of these is just $P(B_{1\times 2}) = 1 + 2x$ and the other is 

$= (1+2x)+x(1+x) = 1+3x+x^2$, which you may easily check by hand. This gives 

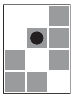
$(1 + 2x)(1 + 3x + x^2) + x(1 + 2x) = 1 + 6x + 9x^2 + 2x^3$.

Meanwhile, returning to our original board, we still must calculate 

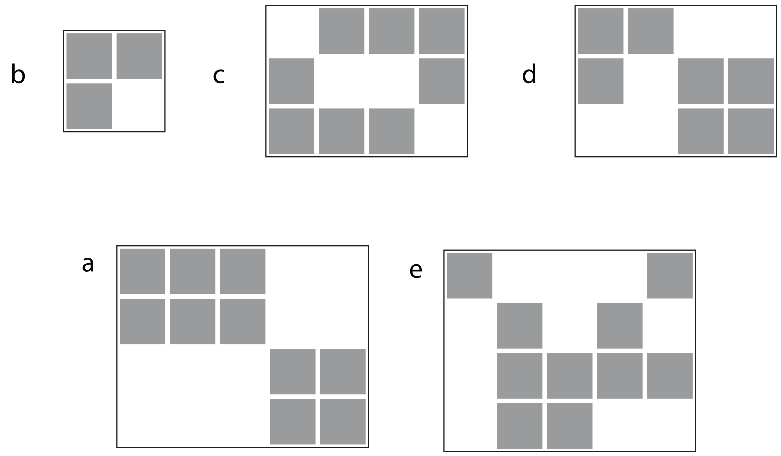
(since we may rearrange rows or colums to make things easier for ourselves).

We now have  $= (1 + 3x + x^2) + x(1 + 2x) = 1 + 4x + 3x^2$

Completing the problem we have

$P($  $) =$ $(1 + 6x + 9x^2 + 2x^3) + x(1 + 4x + 3x^2) = 1 + 7x + 13x^2 + 5x^3.$

9. Find the rook polynomials for the following boards. (Hint: be strategic!)



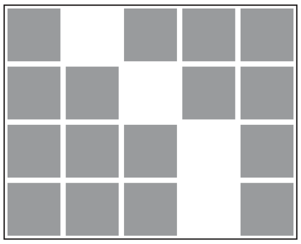
10. In the environment of your choice, program the recursive calculation of the rook polynomial of a board, and check your answers to all of the above problems.

P.I.E. and complementary boards

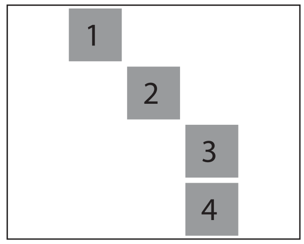
Before continuing, recall from Exercise 7 that we have a formula for $P(B_{m\times n})$. We'll need these rook coefficients, so let us define

$$P(m, n, k) := [x^k]P(B_{m\times n}) = \underline{\qquad\qquad}$$

On the left is a complicated board B. It will be tedious to calculate its rook polynomial by hand. It is easy to calculate the rook polynomial for its complimentary board B', shown on the right: $P(B') = P(B_{1\times 1})^2 P(B_{1\times 2}) = (1+x)^2(1+2x) = 1 + 4x + 5x^2 + 2x^3$.



a



b

We know the rook coefficients for B' but we want the number of ways to place k rooks on B, for each k.

We approach this by first considering out all possible ways to place k rooks on the complete board, $B_{4\times 5}$. From Exercise 7, you'll recall this number, which we are denoting $P(4,5,k)$. $P(4,5,3) = 240$

Out of these ______ ways to place k rooks on $B_{4\times 5}$, let's define conditions we *don't* want to count:

- N_a, the number of placements with a rook on square (a)
- N_b, the number of placements with a rook on square (b)
- N_c, the number of placements with a rook on square (c)
- N_d, the number of placements with a rook on square (d)

Fixing a specific k, N_0 is the number of ways that at least *none* of these conditions are satisfied. In other words, $N_0 = P(4,5,k)$.

We can count $N_a, N_b, \ldots$: For example, a placement is counted in N_a if there is a rook on square (a) and then $(n-1)$ rooks are placed on the $(4-1)\times(5-1)$ subboard disjoint from square (a) (with no regard of whether or not any of the other special squares end up with a rook). This is the same for each term, and there are exactly r_1 terms so we have

$$N_1 = N_a + N_b + N_c + N_d = r_1 \cdot P(4-1, 5-1, k-1)$$

We can count $N_2 = N_{ab} + N_{ac} + \cdots + N_{cd}$, but some of these terms are 0. Some pairs of conditions can't be satisfied. For example, there is no way to place rooks on both (c) and (d) and so $N_{cd} = 0$.

But we do know that there are exactly r_2 terms, the number of ways two rooks may be placed on B'. Each term is the number of ways to place two rooks on the specified squares, and the other $(n-2)$ rooks on a $(4-2) \times (5-2)$ board. In other words

$$N_2 = r_2 P(4-2, 5-2, k-2)$$

In the same way we can work out $N_3 = r_3 P(4-3, 5-3, k-3)$.

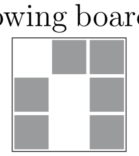
As a check, give $N_4 =$ ____________________. What is N_5?

With $N_0, N_1, \ldots$ in hand, we can calculate E_0, the number of ways exactly none of the conditions are satisfied and that all k rooks are placed on the original board B.

Theorem Let B be an $m \times n$ board. Let $r_0, r_1, \ldots$ be the rook coefficients of the complementary board B', so $P(B') = r_0 + r_1 x + r_2 x^2 + \ldots$. For each $k \in \{0, 1, 2 \ldots\}$, the rook coefficient R_k of the original board satisfies:

$$R_k = r_0 P(m, n, k) - r_1 P(m-1, n-1, k) + \cdots = \sum_j (-1)^j r_j P(m-j, n-j, k-j)$$

Notice that the bounds of the sum at the right don't need to be specified — all but finitely many of the terms are zero.

11. Fill in the proof of the theorem.
12. For the example above, work out
    (a) R_0, R_1 (we have easy ways to do those!)
    (b) R_2,
    (c) R_3,
    (d) R_4.
13. For the following board B, find $P(B)$ directly, and then as a check calculate its rook coefficients using P.I.E. 
14. Mrs. Teabottle is quite worried about the annual Gardenia Banquet. Ms. Aggravant, Mr. Bellicose, Mrs. Colic and Dr. Dreck all detest each other and will have to sit at different tables. Fortunately, there are five tables (Red, Green, Blue, Yellow and Orange) to choose from.

But neither Ms. Aggravant and Mr. Bellicose may sit at the Red or Green Tables. Mrs. Colic absolutely cannot be allowed to go near the Yellow or Orange tables (we all remember what happened last year!) and Dr. Dreck must not sit at the Blue or Yellow tables.

Oh dear! cries Mrs. Teabottle.

But she worries too much! How many ways can she seat these difficult guests?

6 Exponential generating functions

Ordinary generating functions are useful for counting things like

- The number of partitions of a number into summands
- The number of solutions to $a + b + c + \cdots = n$
(i.e. the number of ways to choose a pile of n `a`'s, `b`'s, `c`'s ...).
- The number of ways to distribute n equivalent objects into "bins"
(say, n donuts to some children).

In each of these, the generating function is the product of polynomials. The logic in the counting is exactly captured by the algebra of polynomial multiplication, and we can even encode constraints on the summands/solutions/bins very simply, just by the powers we use in the polynomials we multiply.

Exponential generating functions are useful for counting things like

- The number of partitions of a set into subsets
- The number of n-letter strings of `a`'s, `b`'s, `c`'s, ...
(i.e the number of ways to distribute the numbers 1st, 2nd, ... nth among the bins `a`'s, `b`'s, `c`'s, ...).
- The number of ways to distribute n different objects into bins
(say, n rare coins to some children, or n students into classrooms).

The basic reason will be the same as for ordinary generating functions: the logic of multiplying exponential generating functions will capture the counting correctly.

An **exponential generating function** for the sequence $a_0, a_1, \ldots$ is

$$a_0\left(\frac{x^0}{0!}\right) + a_1\left(\frac{x^1}{1!}\right) + a_2\left(\frac{x^2}{2!}\right) + \cdots = \sum_{k=0}^{\infty} a_k\left(\frac{x^k}{k!}\right)$$

We'll write $(a_k) \overset{\text{egf}}{\longleftrightarrow} A(x)$ to mean that $A(x)$ is the exponential generating of the sequence (a_k).

In other words, $(a_k) \overset{\text{egf}}{\longleftrightarrow} A(x)$ if and only if each $a_k = [x^k/k!]A(x)$

Before we see the utility of these, let's try some warm up exercises; keep in mind that

$$r^0 + r^1 + r^2/2 + r^3/3! + r^4/4! + \cdots = e^r$$

1. Find the exponential generating function for the sequences
   a) $1, 1, 1, 1, 1, ...$
   b) $0, 0, 1, 1, 1, 1, \ldots$
   c) $1, 2, 4, 8, 16, 32, ...$
   d) $1, -1, 1, -1, 1, -1, \ldots$
   e) $1, 0, 1, 0, 1, 0, 1, 0, ...$
2. Determine the sequence generated by the following exponential generating functions:
   a) e^{3x}
   b) e^{-x^2}
   c) $2xe^{3x}$
   d) $2xe^{3x} - e^{-x} - x^2$
   e) $\dfrac{1}{1-x}$ (as an exponential generating function!)
3. Write the following series in closed form:
   a) $x + x^2/2 + x^3/3! + x^4/4! + \ldots$
   b) $x^2 + x^3 + x^4/2 + x^5/3! + x^6/4! + \ldots$
   c) $1 - x^2/1! + x^4/2! - x^6/3! + x^8/4! + \ldots$
   d) $1 - x^2/2! + x^4/4! - x^6/6! + x^8/8! + \ldots$
   e) $1 + 3x + 9x^2/2 + 27x^3/3! + 81x^4/4! + \ldots$
   f) $(1 + x + x^2/2 + x^3/3! + \ldots) \cdot (1 + x + x^2/2 + x^3/3! + \ldots)$
   g) $(1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \ldots) \cdot (1 - x + x^2/2 - x^3/3! + \ldots)$
4. Show $(\sum_k x^k/k!)^n = \sum_k (nx)^k/(k!)$

How to calculate with exponential generating functions

Let's first recall the number of strings of the letters `M I S S I S S I P P I` is $\frac{11!}{1!4!4!2!} = 34650$, because there are 11! strings of 11 distinct letters, but this overcounts by the 1! way to rearrange the one `M`, 4! ways to rearrange the four `S`'s, 4! ways to rearrange the four `I`'s and 2! ways to rearrange the pair of `P`'s. As a quick test, how many ways are there to rearrange the letters in
`S P I M I S M I S M S` ? 46200

Suppose instead we want to count *all possible* 11-letter strings using the letters `M`, `I`, `S`, `P`.

In fact, we already know the answer to this: there are 4^{11} ways. And if we let a_n be the number of n-letter strings, then $a_n = 4^n$ and the exponential generating function is

$$(4^n) \overset{\text{egf}}{\longleftrightarrow} A(x) := 4^0 + 4^1x + 4^2x^2/2! + 4^3x^3/3! + \cdots + (4x)^n/n! + \cdots = e^{4x}$$

Let's see this another way!

If an n-letter string has m `M`'s, i `I`'s, s `S`'s and p `P`'s, we know, first off, that $m + i + s + p = n$, and each $m, i, s, p \geq 0$. For a given m, i, s, p, the number of such strings is $\frac{n!}{m!i!s!p!}$, and so the sequence is (not so usefully)

$$a_n = \sum_{\substack{m+i+s+p=n \\ m,i,s,p \geq 0}} \frac{n!}{m!i!s!p!}$$

(btw: we know how many possibilities there are for m, i, s, p: $(-1)^4\binom{-4}{n} = \binom{n+3}{3}$)

But let's build this up.

The exponential generating function for the number of n-letter strings of just `M`'s is $M(x) := 1 + x + x^2/2 + x^3/3! + \cdots + x^m/m! + \cdots = e^x$ (there is exactly one n-letter string of just `M`'s).

Similarly the exponential generating functions for strings of just `S`'s is $S(x) := \sum x^s/s! = e^x$, of just `I`'s is $I(x) := \sum x^i/i! = e^x$ and of just `P`'s is $P(x) := \sum x^p/p! = e^x$.

We already knew the number a_n of n-letter strings with all four letters is 4^n, with egf $A(x) = e^{4x}$ —and it isn't coincidence that $A(x) = M(x)\ I(x)\ S(x)\ P(x) = (e^x)^4$!

Let's take a close look at

$$(1+x+x^2/2+x^3/3!+\cdots+x^m/m!+\dots)(1+x+x^2/2+x^3/3!+\cdots+x^i/i!+\dots)$$

$$(1+x+x^2/2+x^3/3!+\cdots+x^s/s!+\dots)(1+x+x^2/2+x^3/3!+\cdots+x^p/p!+\dots)$$

What is the coefficient of, say, $x^{11}/11!$? Each term that contributes to this will come from a choice of x^m from the first series, x^i from the second, x^s from the third and x^p from the last, with $m+i+s+p=11$, each $m,i,s,p\geq 0$. Let's consider the specific case $m=1, i=4, s=4, p=2$:

$$\ldots+\left(\frac{x^1}{1!}\right)\left(\frac{x^4}{4!}\right)\left(\frac{x^4}{4!}\right)\left(\frac{x^2}{2!}\right)+\dots=\cdots+\frac{1}{1!4!4!2!}\,x^{11}+\dots$$

But we want the coefficient of $x^{11}/11!$:

$$=\cdots+\frac{11!}{1!4!4!2!}\,x^{11}/11!+\dots$$

That is exactly the correct number of 11-letter strings formed from one `M`, four `I`'s, four `S`'s and two `P`'s!!

The coefficient of $x^{11}/11!$ will be the sum over all possible numbers of m of `M`'s, i of `I`'s, s of `S`'s and p of `P`'s, with m,i,s,p summing to 11 and each non-negative — and in general

$$[x^n/n!](e^x)^4=\sum_{\substack{m+i+s+p=n\\ m,i,s,p\geq 0}}\frac{n!}{m!i!s!p!}$$

It works!!

Furthermore, suppose we put restrictions on the numbers of each letter. Perhaps, for example, we want

• an *even* number of each letter. Then as before, we simply include the summands that encode the restrictions. The exponential generating function for the numbers of n-letter strings with an *even* number of `M`'s, `I`'s, `S`'s and `P`'s is

$$(1+x^2/2!+x^4/4!+x^6/6!+\dots)^4=\cosh^4 x$$

• If we require at least one of each letter, the egf is $(e^x-1)^4$.

• If we want n-letter strings taken from the specific letters `M I S S I S S I P P I` we have $m+i+s+p=n$, but also $0 \le m \le 1$, $0 \le i, s \le 4$ and $0 \le p \le 2$ and the generating function is

$$(1+x)(1+x+x^2/2+x^3/3!+x^4/4!)^2(1+x+x^2/2)$$

It's not so easy to work with this directly: use Wolfram Alpha or something similar to expand and actually find the coefficients of each $x^n/n!$. Or we can use P.I.E.

More Exercises:

5. Ms. Witham will distribute n pennies to Alice, Bill, Carl, Dora and Edward; Alice and Bill (her favorites) will get at least five pennies, and all the others will get at least one penny. But Edward (the rascal!) will get no more than three.
   - How many ways may she distribute $n = 20$ pennies?
   - Write, in closed form, an ordinary generating function for the number of ways that n pennies may be distributed.
6. Give an exponential generating function for the number of ways Ms Witham can distribute n rare coins, all different, Alice and Bill still receiving at least five each, the other children at least one, Edward receiving three or fewer.
7. Give the exponential generating function for the number of n-letter strings formed from
   - the letters `A B C D R`
   - the letters `A B C D R` if an odd number of each letter must be used.
   - the letters `A B C D R` if no more than three of each letter may be used. Use Wolfram alpha to obtain the exact numbers as coefficients of $x^n/n!$.
   - the letters
     `A B R A C A D A B R A`.
8. A company assigns n employees into five teams. Give an exponential generating function for the number of ways that this may be done if
   a) Every team has at least two employees.
   b) Every team has no more than ten employees.
9. Prove a curious identity: $k^n = \displaystyle\sum_{\substack{c_1+c_2+\cdots+c_k=n \\ c_i \geq 0}} \frac{n!}{\prod(c_i)!}$.

7 Recurrence Relations

A recurrence relation on a sequence $a_0, a_1, ...$ is an expression of each a_n in terms of earlier values in the sequence, together with some *initial terms*. The most famous example of a recurrence relation is that defining the Fibonacci numbers:

$$f_0 := 0, f_1 := 1, \text{ and for each } n \geq 2, f_n := f_{n-1} + f_{n-2}$$

From this, we can compute the sequence: 0,1,1,2,3,5,8,13,21,34,...

But what is, say, f_{100}, or f_{100000}? How fast do these grow? We seek a *closed form* expression for the values f_n, a formula in terms of n itself.

We will be concerned with *linear* recurrence relations *with constant coefficients*, particularly *homogeneous* ones.

Here are some simple examples we can quickly solve by hand:

a) A savings account earns 1% interest compounded annually. Let a_0 be the initial amount in the account and for each $n \in \mathbb{N}$, let a_n be the amount in the account after n years.

   a) If you know a_{n-1}, what is a_n? (What is a recurrence relation for a_n in terms of a_{n-1}?)

   b) Starting with a_0, using the recurrence relation to find a_1, a_2, etc, what is the closed form for a_n?

b) (The same thing) One thousand liters of preservative are in an indusrtrial tank. Each minute one third of the remaining preservative is removed. Let s_n be the amount of preservative remaining after n minutes. Give a recurrence relation for s_n in terms of s_{n-1}, and give the closed form for s_n.

c) (The same thing) A calculation processes a string of length n and solves some sort of puzzle in T_n seconds. Each additional letter doubles the amount of time the calculation takes. Give a recurrence relation on T_n in terms of T_{n-1}, and a closed form expression for T_n in terms of T_0.

7.1 Homogeneous linear recurrences with constant coefficients

Rather than stating the general theorem for homogeneous linear recurrence relations, we'll use examples:

Suppose

$$a_n = a_{n-1} + 6a_{n-2}, \text{ with initial conditions } a_0 = 1, a_1 = 4$$

The sequence begins 1, 4, 10, 34, 94, 298 ... But what is the 100th term? A formula for the nth term? How fast do these grow?

We rewrite the recurrence as

$$a_n - a_{n-1} - 6a_{n-2} = 0$$

The *characteristic polynomial* of this relation is

$$x^2 - x + -6 = 0$$

which factors as $x^2 - x + -6 = (x-3)(x+2)$ with roots $x = 3, -2$.

The closed form solution to the relation will be

$$a_n = A\ 3^n + B\ (-2)^n$$

with A and B depending on the initial conditions of the relation $a_0 = 1, a_1 = 4$

When $n = 0$, we have $a_0 = 1 = A\ 3^0 + B(-2)^0 = A + B$.

When $n = 1$ we have $a_1 = 4 = 3A - 2B$. Solving the system of equations

$$\begin{aligned} 1 &= A + B \\ 4 &= 3A - 2B \end{aligned}$$

we find that $A = 6/5$ and $B = -1/5$. We thus have

$$a_n = \frac{6}{5}\ 3^n - \frac{1}{5}\ (-2)^n$$

Checking the first few values we verify that this is correct!

This process works for any homogeneous linear recurrence relation *for which the roots of the characteristic polynomial are distinct.*

1. Suppose $b_n = b_{n-1} + 2b_{n-2}$ with $b_0 = 3, b_1 = 0$. Give a closed form expression for b_n.
2. Suppose $c_n = 4c_{n-2}$ with $c_0 = 4, c_1 = 4$. Give a closed form expression for c_n.
3. (higher degree) Suppose $d_n = 2d_{n-1} + 5d_{n-2} - 6d_{n-3}$ with $d_1 = 1, d_2 = 4$ and $d_3 = 10$. Give a closed form expression for d_n.
4. (irrational roots) Suppose $e_n = 2e_{n-1} + e_{n-2}$ with $e_0 = 0$ and $e_1 = 1$. Give a closed form expression for e_n.
5. What is a closed form for the Fibonacci numbers?

But if a root is repeated, we must add additional terms to our general solution. Again, we demonstrate by examples:

Suppose

$$a_n = 6a_{n-1} - 9a_{n-2}, \quad a_0 = 1, a_1 = 9$$

Our polynomial is $x^2 - 6x + 9 = (x-3)^2$ and the root $x = 3$ is repeated twice. The general solution will be of the form $a_n = (An + B)\, 3^n$, and we solve for A and B as before, using our initial conditions.

Similarly, for $b_n = 9b_{n-1} - 27b_{n-2} + 27b_{n-3}$ the characteristic polynomial is $x^3 - 9x^2 + 27x - 27 = (x-3)^3$; the root $x = 3$ is repeated three times, and the general solution will be of the form $a_n = (An^2 + Bn + C)\, 3^n$, and once our initial conditions are specified, we could solve for A, B, C.

6. Solve $a_n = 4a_{n-1} - 4a_{n-2}$ with $a_0 = 4, a_1 = 6$.
7. Solve $b_n = 3b_{n-2} + 2b_{n-3}$ with $a_0 = 3, a_1 = -1, a_2 = 8$. (You can use Wolfram Alpha or some other symbolic calculator to help with the factoring).

8. Here are some more for practice:
   (a) $a_n = 6a_{n-1} - 8a_{n-2}$, $a_1 = 6$, $a_2 = 16$.
   (b) $b_n = b_{n-1} + 2b_{n-2}$, $b_0 = 2, b_1 = 1$.
   (c) $c_n = 2c_{n-1} - 2c_{n-2}$ with $c_0 = -2, c_1 = -1$. This has complex roots, but it works in just the same way.
   (d) $d_n = 6d_{n-1} - 9d_{n-2}$, with $d_{-1} = 0, d_0 = 2$.
   (e) $e_n = 2e_{n-2} + 3e_{n-4}$, $e_0 = 5, e_1 = 0, e_2 = 11, e_3 = 0$.
   (f) $f_n = 4f_{n-1} - 4f_{n-2}$ with $f_0 = 4, f_1 = 8$.
   (g) $g_n + 5g_{n-1} + 8g_{n-2} + 4g_{n-3} = 0$, $g_0 = g_1 = g_2 = 0$.

9. Give a recurrence relation, including initial conditions, for s_n, the number of strings of the letters `0` and `1` in which no two `0`'s are adjacent.

10. Give a recurrence relation, including initial conditions, for r_n, the number of strings of the letters `A`, `B`, `C` where the substrings `AC` and `CA` do not appear. Using the recurrence relation, how many such strings of length 10 are there?

11. Let s_n be the number of ways to select squares from a $2 \times n$ grid, so that the selected squares form a connected region, reaching both ends of the grid. As discussed, $s_n = 2s_{n-1} + s_{n-2}$.
    (a) Solve the recurrence relation (you will need the initial conditions).
    (b) Use the closed formula you found in (a) to find the exact integer value of s_{16}.[21]

12. (a) Give a recurrence relation for the number of ways t_n to fill a $2 \times n$ grid with $2 \times 1, 1 \times 2$ and 2×3 blocks, including initial conditions.
    (b) Using the recurrence relation, how many ways are there to tile a 2×10 grid?

7.2 Non-homogeneous linear recurrences with constant coefficients

A *non-homogeneous* linear recurrence relation has, in addition, some function of n, such as:

$a_n = 2a_{n-1} + 3^n, a_0 = 6$

$b_n = 3b_{n-1} + 4b_{n-2} + n, b_1 = 2, b_2 = 5$

$c_n = 4c_{n-2} + 1, c_0 = 0, c_1 = 3$ (1 is a function of n— the constant function!)

$d_n = 4d_{n-1} - 4d_{n-2} + 2^n, d_0 = 1, d_1 = 4$

The idea is that our solution will have two pieces: the solution to the original homogeneous recurrence, and an additional, "particular" term that that looks similar to the additional term. For example, for a_n, the homogeneous recurrence is denoted $a_n^{(h)} = 2a_n^{(h)}$, with a solution of the form $a_n^(h) = A2^n$. The additional term is 3^n and so we guess a particular solution $a_n^{(p)} = B3^n$.

We try to solve for the particular solution first, ignoring the initial conditions.

$a_n^{(p)} = 2a_{n-1}^{(p)} + 3^n$

[21]You will need to take powers of numbers of the form $a + \sqrt{b}$; work out $(a + \sqrt{b})^2$, then $((a + \sqrt{b})^2)^2$, etc.

$B3^n = 2B3^{n-1} + 3^n$. Plugging in $n = 1$ we have

$3B = 2B + 3$ and so $B = 3$, and $a_n^{(p)} = 3 \cdot 3^n = 3^{n+1}$.

Our general solution is of the form $a_n = a_n^{(h)} + a_n(p) = A2^n + 3^{n+1}$.

Finally, we use the initial condition $a_0 = 6$ to solve for A. We have $6 = A2^0 + 3^1$ and so $A = 3$. Finally $a_n = 3 \cdot 2^n + 3^{n+1}$.

If the additional term is

- a constant, guess an unknown constant;
- a polynomial, guess an unknown polynomial of the same degree;
- an exponential b^n, guess an unknown multiple of that exponential — if b is not already a root of the char. polynomial.
- a polynomial multiple $p(n)b^n$, guess an unknown polynomial multiple of the same degree,
- an exponential b^n where b is a root, guess a multiple of $n^k b^n$ where k is the multiplicity of the root, [22]
- a polynomial multiple $p(n)b^n$ where b is a root, guess a polynomial multiple, bumping up the degree. [23]

13. Solve the recurrences given above, filling in your own initial conditions.
14. Solve the following recurrence relations; you should try out a few terms by hand, to be able to check your solution.
    (a) $a_n = 5a_{n-1} - 6a_{n-2} + n$, $a_0 = 2$, $a_1 = 5$.
    (b) $b_n = 5b_{n-1} - 6b_{n-2} + 4^n$, $b_0 = 2$, $b_1 = 5$.
    (c) $c_n = 5c_{n-1} - 6c_{n-2} + 3^n$, $c_0 = 2$, $c_1 = 5$.
    (d) $d_n = 5d_{n-1} - 6d_{n-2} + 1 + 3^n$, $d_0 = 2$, $d_1 = 5$.
15. Ned pays $150 per month on a credit card loan of $10,000, with an annual interest rate of 24% compounded monthly (so 2% per month, or 26.824% APR). Let p_n be the amount he owes on the card after n months.

[22] Don't be fooled if the additional term is a constant, b, which also a root. So what! We are not in this special case, with an additional term of the form b^n. The additional term is just a constant, so guess a constant.

[23] On the other hand, there is a sneaky sub-case if one of the roots is $b = 1$. Any additional term $P(n)$ is actually of the form $P(n)1^n$, and the additional term actually *is* in this special case. The degree of our particular solution will be increased by the multiplicity of the root 1.

(a) Give a recurrence relation for p_n.

(b) Solve this recurrence relation.

(c) After how many months, if ever, will Ned pay off the debt?

(d) Changing the payment, what is the *maximum* payment that Ned can make and *never* pay off the debt? (What payment keeps the principal constant)?

(e) What should the payment be to pay off the debt within 36 months?

16. Consider the number s_n of strings of length n from letters `A`, `B`, `C`, in which there are an odd number of `A`'s. The strings of length n with an odd number of A's either end in a `B` or `C`, and are therefore twice as numerous as those of length $(n-1)$; or end in an `A` and therefore correspond to the strings of length $(n-1)$ with an even number of `A`'s. We can count those as the total number of strings, less the ones with an odd number of `A`'s. Therefore $s_n = 2s_{n-1} + (3^{n-1} - s_{n-1})$ or
$$s_n = s_{n-1} + 3^{n-1}$$
Solve this recurrence relation.

17. The recurrence $a_n = a_{n-1} + n$ with $a_0 = 0$ gives the triangular numbers, $0, 1, 3, 6, 10, .., a_n = (n^2+n)/2, ...$ These are not of the form $a_n = a_n^{(h)} + a_n^{(p)}$ where $a_n^{(h)}$ is some multiple of 1^n and $a_n^{(p)}$ is some guess $Cn + D$. What has gone wrong?

18. Solve the recurrence $a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0$ with $a_0 = 0, a_1 = 1, a_2 = 8$. Calculate a few more terms of the sequence using the recurrence and check your solution is correct.

7.3 Generating functions and recurrence relations

Next, we will see how to obtain a generating function from a recurrence; we've already seen how to work out what sequence a generating function generates, so this gives another way to solve a recurrence relation.

And the reverse holds as well: given a generating function, we can often read out a recurrence relation!

19. Suppose a sequence c_n happens to have generating function $\sum c_n x^n = \dfrac{2x-1}{(x-1)(x+5)}$. Find a formula for c_n: first find values A and B so that $\dfrac{2x-1}{(x-1)(x+5)} = \dfrac{A}{x-1} + \dfrac{B}{x+5}$ — but then these are geometric series, and you know their coefficients. What is each c_n?

20. Similarly, what sequences are generated by these generating functions

(a) $1/(1-x^2)$, in two ways

(b) $\frac{x-7}{x^2+x-2}$

(c) $\frac{x+2}{1+x-2x^2}$

(d) $\frac{1}{1-x-x^2}$

21. Let's find generating functions from recurrence relations. We'll begin with a familiar case:

(a) Let A be the ogf for the sequence defined by the recurrence relation

$$a_n–3a_{n-1} = 0 \text{ with initial condition } a_0 = 4$$

- Write out the series A–$3Ax$, gathering up terms with the same power of x.
- Apply the recurrence relation to simplify A–$3xA$ (dramatically).
- Solve for A, writing A in closed form.

(b) Let B be the ogf for the sequence defined by the recurrence relation

$$b_n–3b_{n-1} + 2b_{n-2} = 0 \text{ with initial conditions } b_0 = 4, b_1 = 3$$

- Write out the series B–$3xB + 2x^2B$, gathering up terms with the same power of x;
- apply the recurrence relation to simplify B–$3Bx + 2Bx^2$;
- and solve for B, writing B in closed form.

(c) Let C be the ogf for the sequence defined by the recurrence relation

$$c_n–3c_{n-1} + 2c_{n-2} + 7c_{n-3} = 0 \text{ with initial conditions } c_0 = 4, c_1 = 3, c_2 = 1$$

- Write out the series $C(1–3x + 2x^2 + 7x^3)$, gathering up terms with the same power of x;
- apply the recurrence relation to simplify $C(1–3x + 2x^2 + 7x^3)$;
- and solve for C, writing C in closed form.

(d) Give a closed form of the generating function of the sequence (z_n) satisfying $c_0a_n + c_1a_{n-1} + ... + c_ka_{n-k} = 0$, with initial conditions $z_0, ..., z_{k-1}$.

22. Here's a nice application of this idea. The Fibonacci numbers satisfy the recurrence $f_n = f_{(n-1)} + f_{(n-2)}$ for $n \geq 2$, with $f_0 = 0, f_1 = 1$. We will prove that for any counting number n,

$$f_n = f_{(n-2)} + f_{(n-3)} + f_{(n-4)} + ... + f_0 + 1$$

(try it!) in the following manner: First find the generating function F for the the Fibonacci numbers using the recurrence. Next, find the generating function for the sequence on the right (fudging the first two terms):

$$0,\ 1,\ 1 + f_0,\ 1 + f_0 + f_1,\ 1 + f_0 + f_1 + f_2,\ ..., 1 + \sum^{n-2} f, ...$$

The generating function of that sequence is the sum of the generating functions of

$$0, 1, 1, 1, 1, ...$$

and the shift of the summation of the Fibonacci sequence.[24] With a little algebra show this is equivalent to F.

23. On the other hand, suppose that a sequence (f_n) has a generating function $F(x) = 3/(1-2x)$. Give a recurrence relation on (f_n). What are the initial conditions on F?

    More generally, suppose $R(x) = P(x)/Q(x)$ where P and Q are polynomials with integer coefficients and the degree of P is less than the degree of Q. What is a recurrence relation on the sequence (r_n) generated by R?

    In fact, the generating function of a sequence is *rational* (in other words, the ratio of two polynomials) if and only if the sequence has a homogeneous linear recurrence relation with constant coefficients— and if and only if the sequence is a linear combination of exponentials of roots of a polynomial. These are very common, and also very special.

24. We can also find the generating function for sequences with *inhomogeneous* linear recurrence relations with constant coefficients. Let's work through an example, finding the generating function for the recurrence

$$a_n - 2a_{n-1} + 3a_{n-2} = n^2, \text{ with } a_0 = 1, a_1 = 3$$

    (a) First, we will need (in closed form) a generating function for the sequence $(q_n) = 0, 1, 4, ..., n^2, ...$ But we've seen this before, in Section 5, as Exercises b) and 15. What is a generating function for the sequence $q_n = n^2$?

    (b) Next, let $f(x) = a_0 + a_1x + a_2x^2 + ... = \sum_0^\infty a_k x^k$ be a generating function for a_n. You will put this in closed form. Since $a_n - 2a_{n-1} + 3a_{n-2} = n^2$, you may write

$$\begin{pmatrix} (a_2 - 2a_1 + 3a_0)x^2 \\ +(a_3 - 2a_2 + 3a_1)x^3 \\ +(a_4 - 2a_3 + 3a_2)x^4 \\ \vdots \end{pmatrix} = \begin{pmatrix} 2^2x^2 \\ +3^2x^3 \\ +4^2x^4 \\ \vdots \end{pmatrix}$$

    On the left side, we are close to having some small polynomial times $f(x)$, and on the right, pretty close to having $Q(x)$. Add in the missing terms, relating $f(x)$ to $Q(x)$, solve for $f(x)$, and give a closed form expression for $f(x)$.

25. Repeating this process for $b_n = b_{n-1} + 3^n, b_0 = 2$, (a) give a closed form generating function and (b) a formula for b_n.

26. Consider the regular language, strings of `A`'s and `B`'s, in which `AB` may not occur, and each string must begin with a `A`. The number w_n of words[25] of length n in the language is exactly $w_n = \texttt{start} \cdot \texttt{transitions}^{(n-1)} \cdot \texttt{end}$ where `start` is the row vector $(1, 0)$; `end` is the column

[24]Remember that we know how the shifting and summation of a sequence affects its generation function.

[25]Ok wait, if you think about this for a moment, you already know this number; but let's watch this process unfold in a case we already understand.

vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; and `transitions` is the 2×2 matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ encoding which letters may follow which. Note that multiplying any matrix by `start` on the left and `end` on the right gives the sum of the first row (try it!).

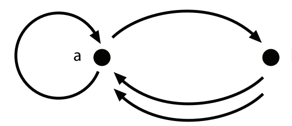
Now for a little magic: The generating function[26]

$$\begin{aligned} f(x) = \sum_0^\infty w_n x^n &= \sum_1^\infty (\texttt{start} \cdot \texttt{transitions}^{(n-1)} \cdot \texttt{end})\ x^n \\ &= x\ \texttt{start} \cdot (\sum_0^\infty \texttt{transitions}^n x^n) \cdot \texttt{end} \\ &= x\ \texttt{start} \cdot (I - x\ \texttt{transitions})^{-1} \cdot \texttt{end} \end{aligned}$$

That's pretty neat! Moreover, the denominator has the same coefficients as a recurrence relation, in reverse order! Taking the inverse of $(I - x\ \texttt{transitions})$:

(a) Give a closed form generating function for $f(x)$, (b) a recurrence relation (with initial values) and
(c) an exact formula for w_n.

In the graph at right, let p_n be the number of paths of length n that start at `A` and end at `B`. Following the same procedure, (a) give a closed form generating function for the number of strings of length n; (b) a recurrence relation using its denominator, and (c) a closed formula for p_n.



[26]In this example, we are counting the number n of vertices we pass through — a path of length $n-1$, why we count these as $\texttt{transitions}^{(n-1)}$ in the first line. In the second line we reindex, pulling an x to the fore, and finally we apply a magic identity, that $\sum A^n x^n = (I - Ax)^{-1}$, so long as the latter is invertible.

8 Pólya enumeration

We want to count distinct colorings up to some symmetry.

For each symmetry g, in a "group" G, and each coloring $c \in C$, let gc be the coloring c transformed by g. Two configurations c and c' are "equivalent up to symmetry" if $c' = gc$ for some $g \in G$. The axioms defining a group assure that this is an equivalence relation[27] and we define equivalence classes $\langle c \rangle = \{c' \mid \exists g \in G \text{ st } gc' = c\}$, the set of configurations equivalent to c under symmetries in G. We can easily count C but wish to count the number N of equivalence classes $\{\langle c \rangle\}$.

For each symmetry $g \in G$, we can let $|\text{fix}(g)|$ be the number of configurations in C that g leaves alone. Then

Burnside's Lemma states that

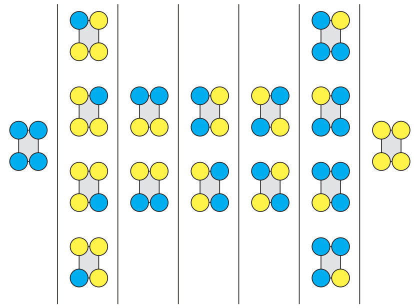
$$N = \frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|$$

Let's use the following example as an application, and to sketch the proof.

Here we want to count the number of ways to color the vertices of a rectangle (let's call them 1,2,3,4) with two colors, perhaps Yellow and Blue, up to symmetry.

It's easy to count the number of colorings, regardless of symmetry, the set C, shown below: there are 2^4 colorings in C. (More generally, if there were k colors and n choices we would have k^n colorings in C.)

[27] There is an identity i for which $ic = c$ for all c, so each c is equivalent to itself. For each $g \in G$ there is an inverse g^{-1} with $g\,g^{-1} = i$; thus if $gc' = c$ for some g, then $g^{-1}c = c'$ and c is equivalent to c' if and only if c' is equivalent to c. Finally, every group G is closed under its operation, here composition. In other words, if g_1, g_2 are symmetries, so too is g_1 followed by g_2, written g_1g_2. Consequently, if c is equivalent to c' and c' is equivalent to c'' then c is equivalent to c'' because there must be g_1, g_2 with $c = g_1c'$ and $c' = g_2c''$, and so $c = (g_1g_2)c''$.

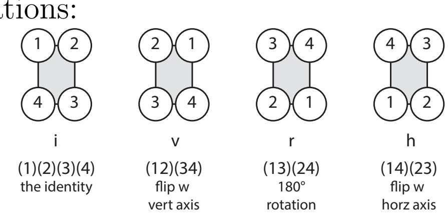


But it is difficult to count the number of colorings that are distinct up to symmetry, and that's our goal. In this example, the colorings C are organized into columns: colorings are in the same column if they are the same up to symmetry. Each column is an equivalence class up to symmetry. We want to count the number of these columns, N— without actually trying to list them!

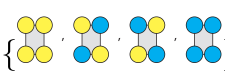
Burnside's Lemma shows the way:

Applying the lemma

The symmetry group of the rectangle has four elements, i, h, r, v: do nothing, flip across a horizontal axis, rotate 180°, flip across a vertical axis, respectively. If we label the vertices 1,2,3,4 as shown, the group acts by these permutations:



For the lemma, we need to understand what are the number of configurations fixed by each group element. For example,

$$\text{fix}(r) = \{$$  $$\}$$

and *every* configuration is fixed by i. It is relatively easy to count the number of elements fixed by a given group element: we really just need to know how many cycles there are of the things we wish to color. Each cycle will have exactly one color, and the colors of the cycles can be chosen independently.

For r there are two cycles: (13) and (24). The vertices 1 and 3 must be the same color and so too must the vertices 2 and 4. There are $k = 2$ colors and $n = 2$ cycles so

$$|\text{fix}(r)| = 2^2$$

Similarly for i there are four 1-cycles, and with two colors, $|\text{fix}(r)| = 2^4$.

- Check that $|\text{fix}(h)|$ and $|\text{fix}(v)|$ are both also 2^2
- Count up $\sum_{g \in G} |\text{fix}(g)|$
- Divide by $|G|$, and obtain N.

1. Answer each of the following. In each, there are several subproblems to take care of:

> - What are the objects to be colored? How many colors are there?
> - What is the symmetry group? How many elements does it have?
> - For each symmetry acting on the objects, how many cycles does it have?
> - Consequently, for each symmetry g, what is $|\text{fix}(g)|$,the number of colorings fixed by g?

How many ways are there to color, up to symmetry,

(a) the vertices of a rectangle with any of three colors? seven colors? 47? 27,637, 1221577

(b) the edges of a rectangle with any of three colors? 36

(c) the vertices and the edges of a rectangle with any of two colors? 84

(d) the vertices of an equilateral triangle with any of three colors? 10

(e) the vertices of a square with any of three colors? 21

(f) the eight vertices, the four short edges and the four long edges of a truncated square, with any of three colors? How many if the vertices are chosen from any of three colors and the edges from any of four colors? 5391522,**

(g) the vertices of a regular pentagon with any of three colors? 39

(h) the vertices of a cube with any of three colors? 333

(i) the faces of a cube with any of three colors? 57

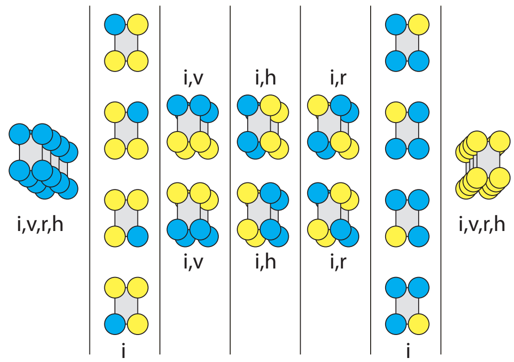
(j) the edges of a cube with any of three colors? 22815

We turn to a

Sketch of a proof of Burnside's Lemma

(A real proof must explicitly use that the symmetries form a "group", a particular, important algebraic structure.) Remember that we can easily count the total number of configurations in C, but cannot easily count their equivalence classes under the action of G, shown as columns in the diagram.

Each coloring in C is "stabilized" under certain elements of G, elements $g \in G$ that do not change c, for which $gc = c$. That is, for each $c \in C$, let $\text{stab}(c) := \{g \in G \mid gc = c\}$. For each c, at least we can be sure that $i \in \text{stab}(c)$, and there may be more besides. In this diagram, for each c, a copy is drawn for each $g \in \text{stab}(c)$.



Notice that there are now $|G|$ configurations in each column, and also that in each column, each of the colorings has the same size stabilizer. In other words, the size $|\langle c\rangle|$ is well-defined and doesn't depend on the representative c.

In other words, we observe (and in general, prove, using group theory [28]), that for each equivalence class

$$|\langle c\rangle| \cdot |\text{stab}(c)| = |G|$$

Consequently, there must be N $|G|$ configurations in the diagram.

But let's count the configurations a different way. Remember that for $c \in C$, and each $g \in \text{stab}(c)$, a configuration is shown. That is, there is a configuration for exactly each of the pairs g, c, $g \in G$, $c \in C$ with $gc = c$. To put this the other way round then, there is a configuration for each $g \in G$, and for each $c \in \text{fix}(g)$.

In other words, dividing through by $|G|$,

$$N = \frac{1}{|G|} \sum_{c \in C} |\text{stab}(c)| = \frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|$$

and we are done.

[28]Proof: Show that stab(c) is a subgroup, and next that there is a well-defined bijection from its cosets to $\langle c\rangle$. Consequently, for each $c' \in \langle c\rangle$, $|\text{stab}(c')| = |\text{stab}(c)|$ and $|\text{stab}(c)| \cdot |\langle c\rangle| = G$

8.1 Cycle index

You have noticed that there is no real difference at all between counting the number of vertices of a rectangle with any of 2 colors versus any of 4 colors, versus any of 12 or any of 481. [29]In each case, the real work is figuring out, for each group element, the pattern of cycles it forms. In the cycle index we write this down as a distinct step, with an aim towards recording more subtle information.

The cycle index will be a kind of generating function, a polynomial P with variables x_1, x_2, ... Each variable x_k will stand in for cycles of length k.

Again let's color the vertices of a rectangle. The identity symmetry i has cycles (1)(2)(3)(4), four 1-cycles, which we record as $x_1 \cdot x_1 \cdot x_1 \cdot x_1 = x_1^4$.

The symmetries h, r, v all happen to have two 2-cycles, and each contribute x_2^2. The cycle index is the defined to be sum of these terms, divided by the number of symmetries in G:

$$P(x_1, x_2) = \frac{1}{4}(x_1^4 + 3x_2^2)$$

This may not look like much of an advance, but it is. Notice first that the number of distinct colorings with two colors is $P(2) = \frac{1}{4}(2^4 + 3\ 2^2) = 7$, found by setting each x_i in P to 2.

But this works in general: with the cycle index in hand, we can instantly answer how many distinct colorings are there, of the vertices of a rectangle by k colors, found by setting each x_i in P to k, giving $P(k) = \frac{1}{4}(k^4 + 3k^2)$.

2. What is the cycle index of
   (a) the vertices of a rectangle?
   (b) the edges of a rectangle?
   (c) the vertices and the edges a rectangle?
   (d) the vertices of an equilateral triangle?
   (e) the vertices of a square?
   (f) the vertices of a regular pentagon?
   (g) the 8 vertices and the 8 edges of a truncated square?

[29]WAIT: We are not counting the number of colorings with exactly two colors, or three, or twelve. In general, P.I.E. will be needed to count these. Here we are counting the number of colorings with *any* of k colors.

(h) the vertices of a cube?

(i) the faces of a cube?

(j) the edges of a cube?

8.2 Pattern inventory

With the cycle index in hand, we can do much more, answering, for example, the number of colorings with specified numbers of colors. (A symbolic calculator, such as Wolfram Alpha or Mathematica will be helpful.)

- Quick, by hand, how many ways are there to color the vertices of a rectangle with one R, two G and one B, up to symmetry?

Our goal is to cook up a generating function, one variable for each color, whose coefficients count colorings. For example, with three colors, say R, G, B, we might use variables r, g, b and look for a polynomial so that the coefficient of $r^i g^j b^k$ will be the number of colorings with i R's, j G's and k B's.

But it is easy!!

In a k-cycle, all of the vertices must be the same color, say R, but that color will therefore appear k times. Replacing x_k with r^k will contribute k r's to the polynomial, representing k R's in the pattern. And that's the entire trick.

For the vertices of the rectangle, the cycle index is $P(x_1, x_2) = \frac{1}{4}(x_1^4 + 3x_2^2)$.

With colors R, G, B, each x_1 (there are four!) will be replaced with one of r^1, g^1, b^1. This is perfectly captured by replacing each x_1 with $(r^1 + g^1 + b^1)$. In the same way each x_2 corresponds to a 2-cycle and will contribute two R's or two G's or two B's, or r^2, g^2, b^2 to the polynomial. This is perfectly captured by replacing each x_2 with $(r^2 + g^2 + b^2)$.

For these three colors, we have the **pattern inventory**

$$P(\sum s^1, \sum s^2, ...) = \frac{1}{4}\left((r^1 + g^1 + b^1)^4 + 3(r^2 + g^2 + b^2)^2\right)$$

where each sum is over the variables for the colors $s = r, g, b$.

Expanding this, we have

$$= 3b^2g^2 + 3b^2gr + b^3g + 3b^2r^2 + b^3r + b^4 + 3bg^2r + bg^3 + 3bgr^2 + br^3 + 3g^2r^2 + g^3r + g^4 + gr^3 + r^4$$

This is a mess, but the coefficients exactly record the number of colorings. For example, there are exactly three ways to color a rectangle with one R, two G's and one B and the coefficient of rg^2b is indeed 3. !!

And setting all of the variables to $r, g, b = 1$, the polynomial sums to 27, exactly $P(3)$, the number of three colorings.

For each, find the pattern inventory and then, using Wolfram Alpha or another symbolic calculator, find how many ways are there to color, up to symmetry,

3. (a) the vertices of a rectangle with two W, two G. 3
   (b) the vertices of a square with one R, two G, one B. 2
   (c) the vertices of a regular pentagon with two R, two G, one B. 4
   (d) the eight vertices of a cube with four W, four G. 7
   (e) the eight vertices of a cube with two W, six G. 3
   (f) the six faces of a cube with any three R, three Y. 2
   (g) the twelve edges of a cube with four each of three colors. 1479
   (h) the 8 vertices and the 8 edges of a truncated square, with four each W, X, Y, Z. 7884456