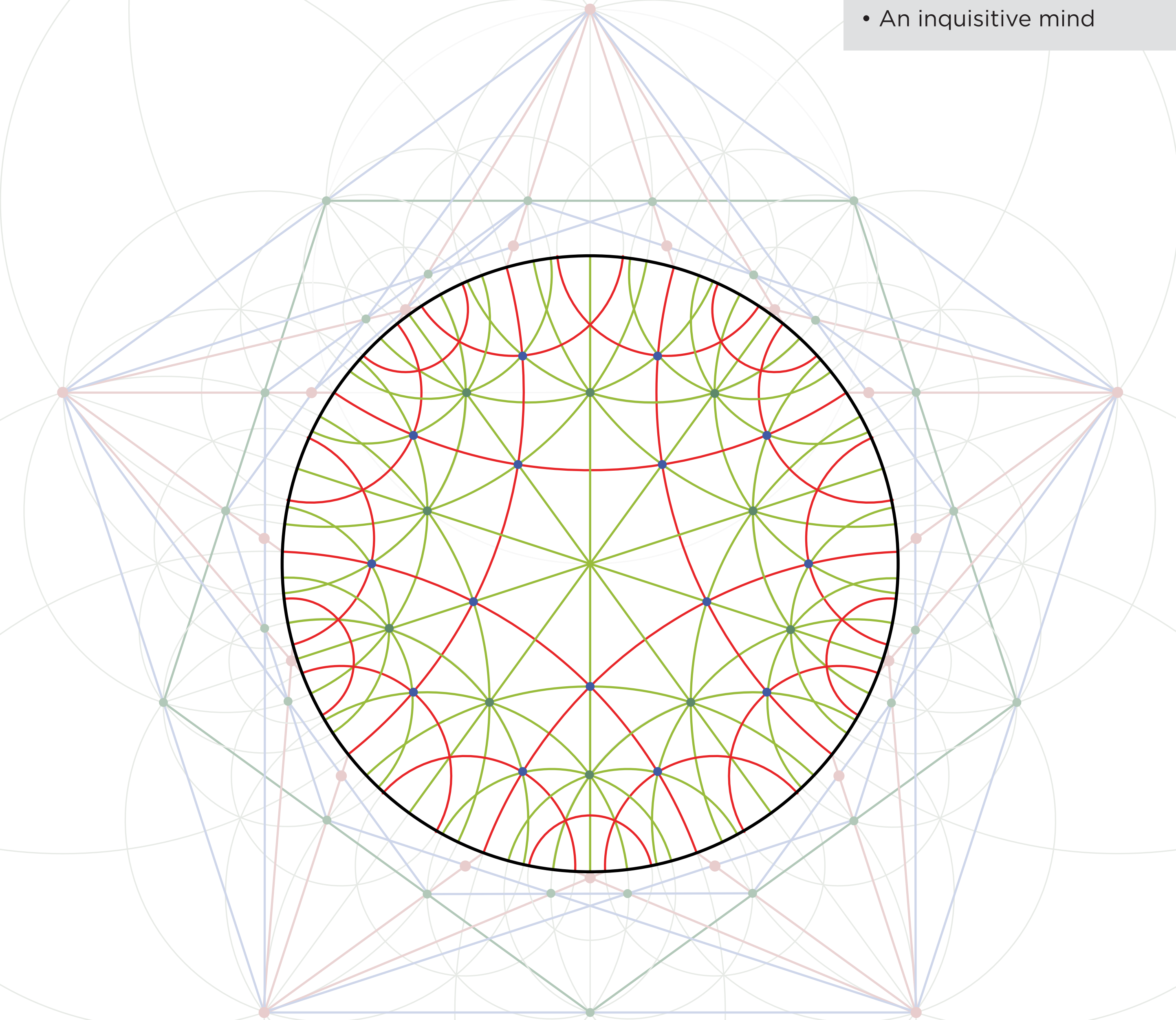
You will need:

- A good, working compass
- A straightedge
- A pad of drawing paper
- Sharp pencils
- Your own printed copy of *Euclid's Elements*
- An inquisitive mind



Euclid's Elements @

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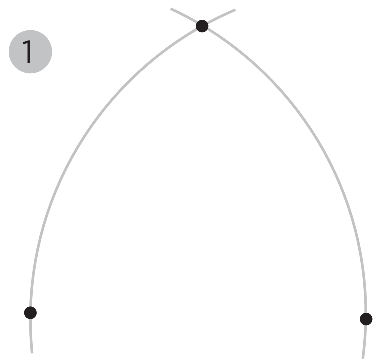
HEY! Draw and write all over these notes!

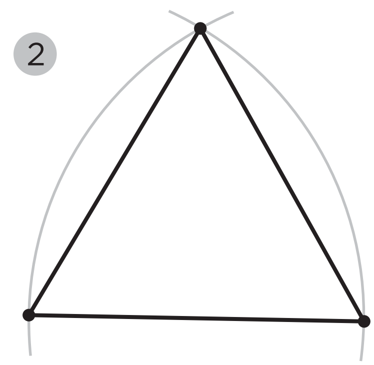
Let’s start right off: How can you construct a perfect equilateral triangle using your straightedge and compass?

Given a pair of points:

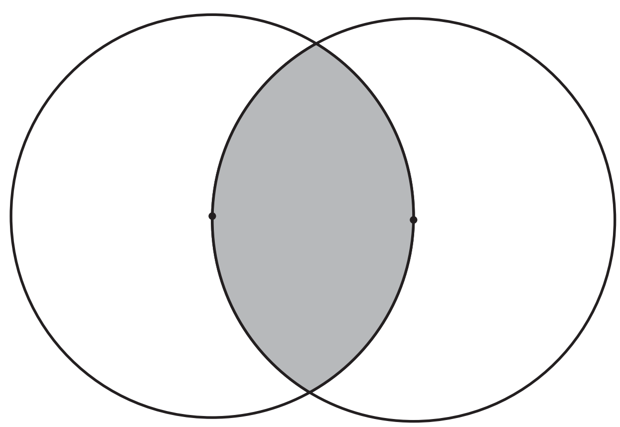


Draw it yourself!

What’s the **proof** this really works?



Something cool to draw while making an equilateral triangle:

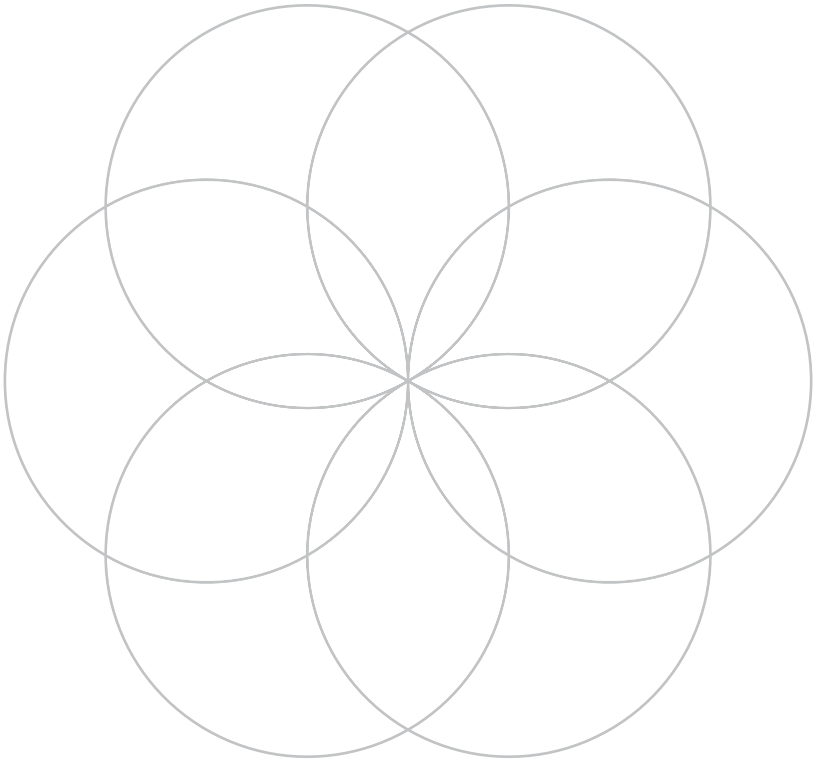


YO! Better keep your notes organized!

This neat drawing comes up when you draw the circles all the way round. It has the very old Latin name of vesica pisces, which means “fish bladder” (I don’t know what a fish bladder is supposed to look like. Do you?)

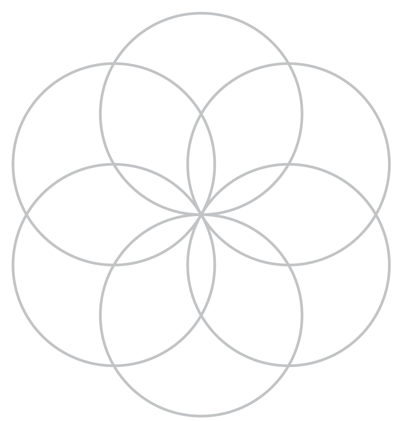
Can you see how to use this to construct a perfect hexagon?

Draw these in your notebook!



Draw more!

Make something cool!



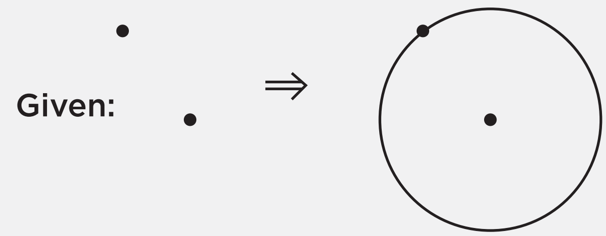
Can you keep on going?

What’s the coolest thing you can construct in your notebook?

Euclid’s Tools

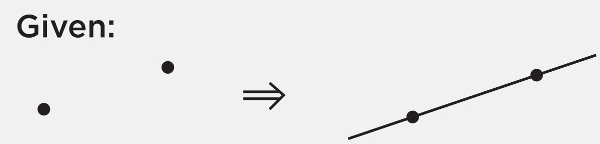
What can you do with a compass?

Given a pair of points, you can draw a circle centered at one, passing through the other.



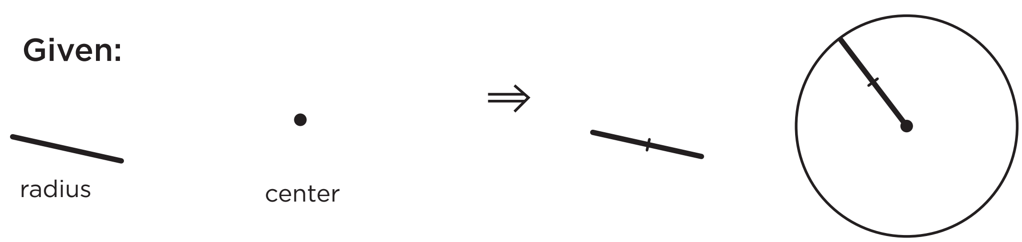
What can you do with a straightedge?

Given a pair of points, you can draw a line, passing through each.



A subtle bonus observation:

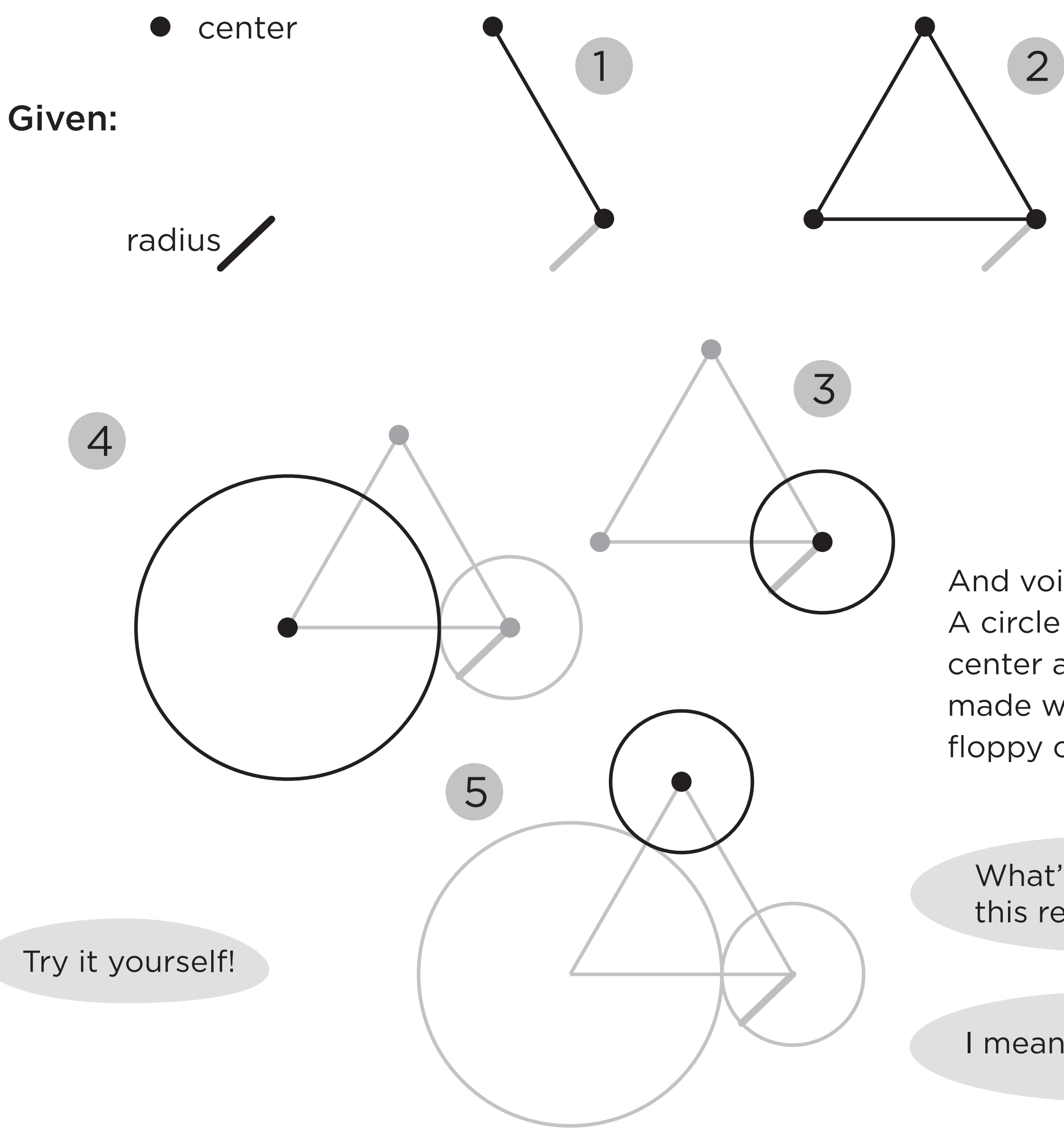
A real compass, of the sort we use today, can do a little more than this. You can measure with it too, setting the radius of your circle and centering it where ever you please.



But Euclid’s compass is a little different: it’s floppy, like a piece of string! Once you pick it up, it collapses. You can only draw a circle if you are given a point that lies on it, just as illustrated in the box at the top of this page.

Does this make a difference? If we had to use a floppy compass would there be things we couldn’t do with a rigid one?

Right off, though, Euclid shows that's just as good! (Book I, Proposition 2) Anything your compass can do, his compass can too, with just a few extra steps. What's really cool is he uses an equilateral triangle to do it! Here's his method:



And voilà!
A circle with the given center and radius, made with Euclid's floppy compass!

Try it yourself!

What's the **proof** this really works?

I mean **always** works?

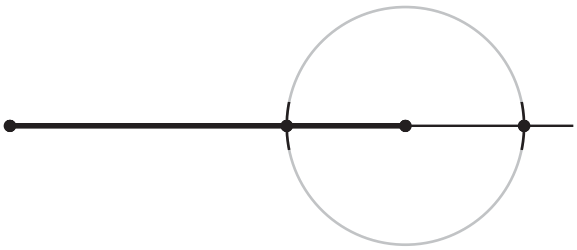
Theorem: Given any center and any radius, Euclid's tools can make a circle with that center and that radius.

A floppy compass is just as good as a rigid compass!

Given:

Step-by-step instructions for constructing a square — but **why do we really obtain a polygon with four equal sides and four right angles?**
What do we need to use to be able to say this?

1



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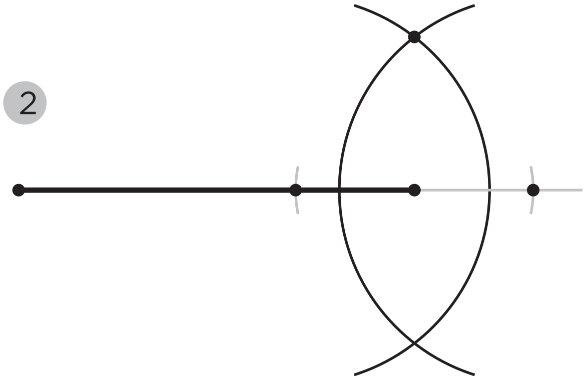

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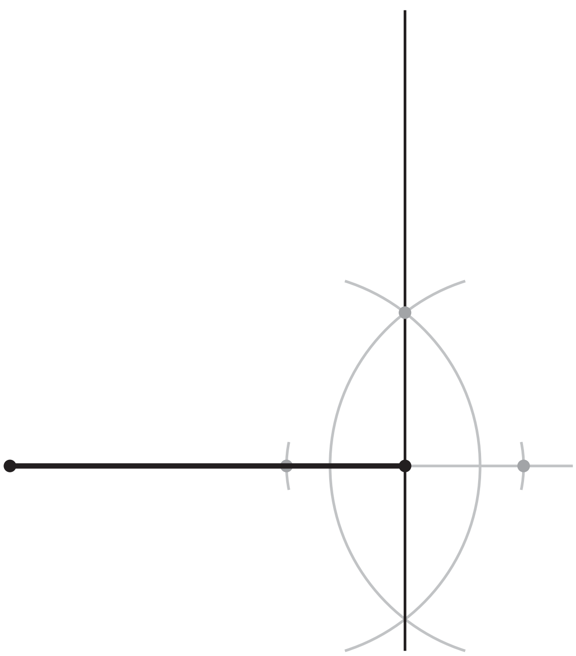

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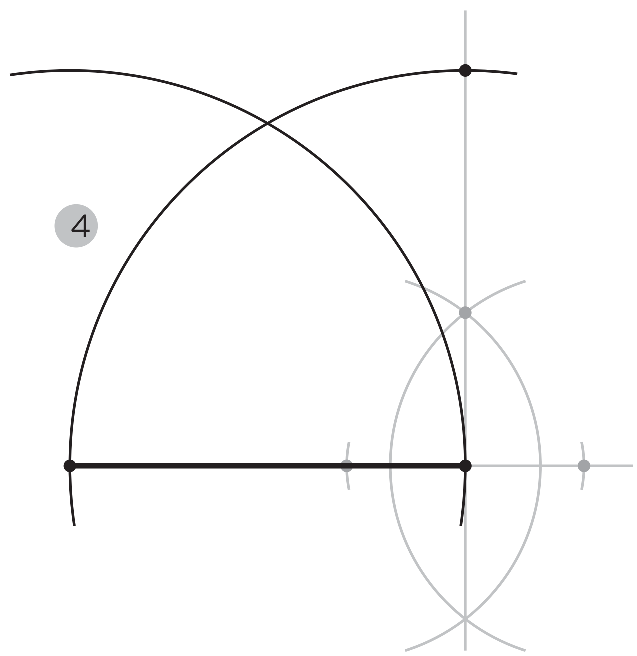

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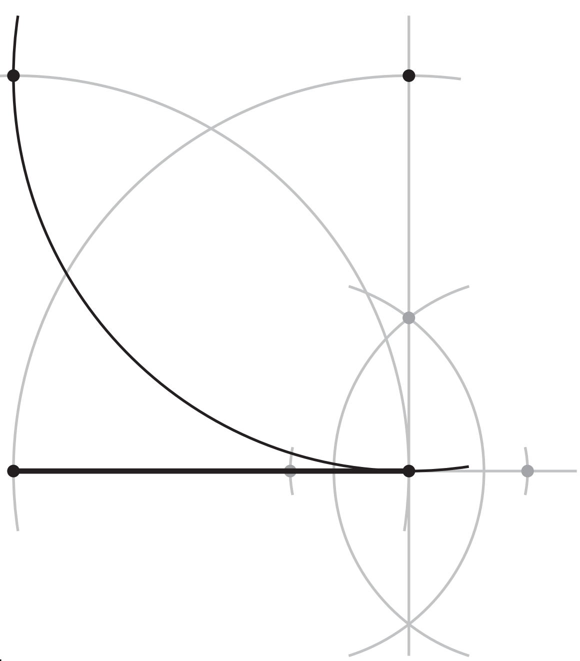

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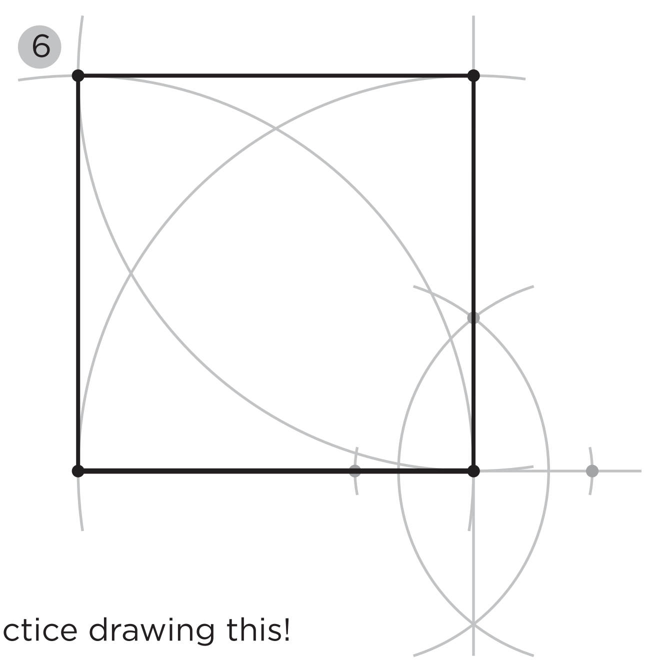
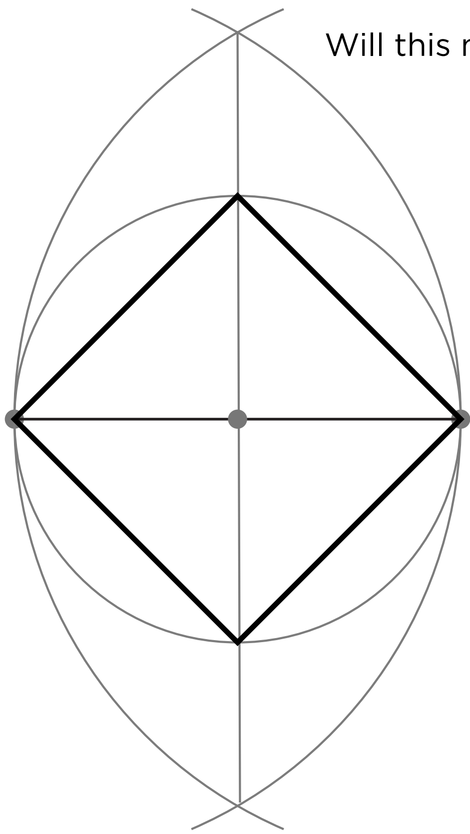


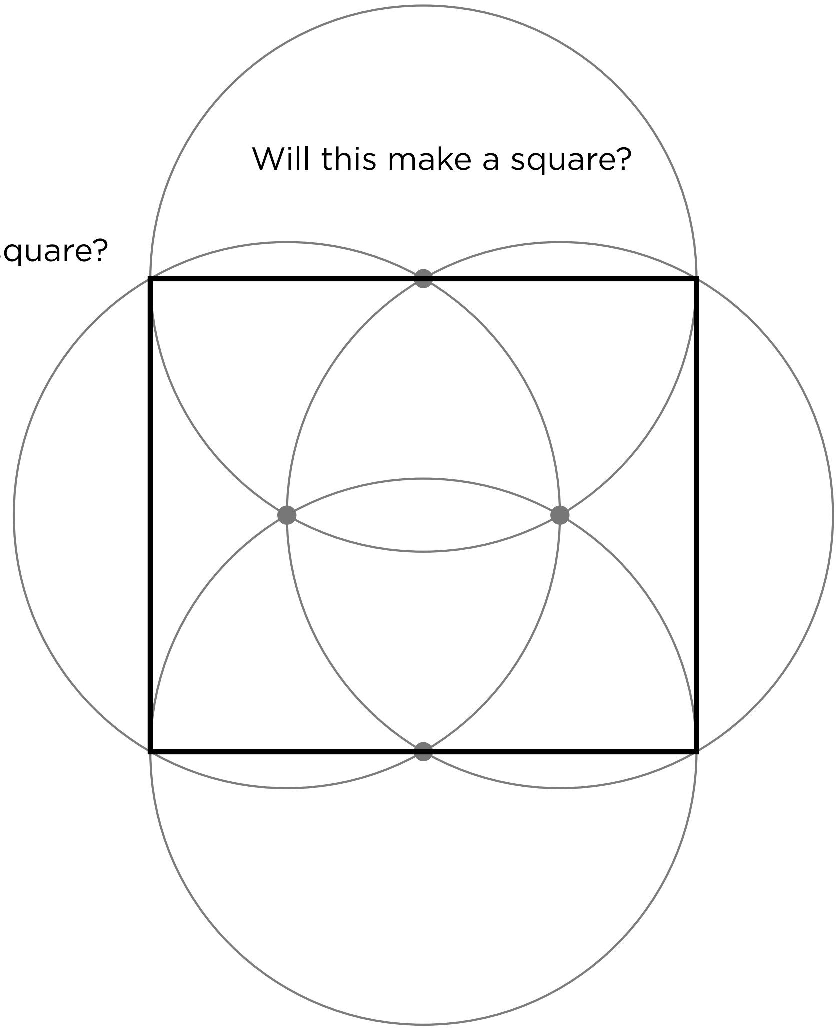
Practice drawing this!

Mysterious Constructions!

Will this make a square?

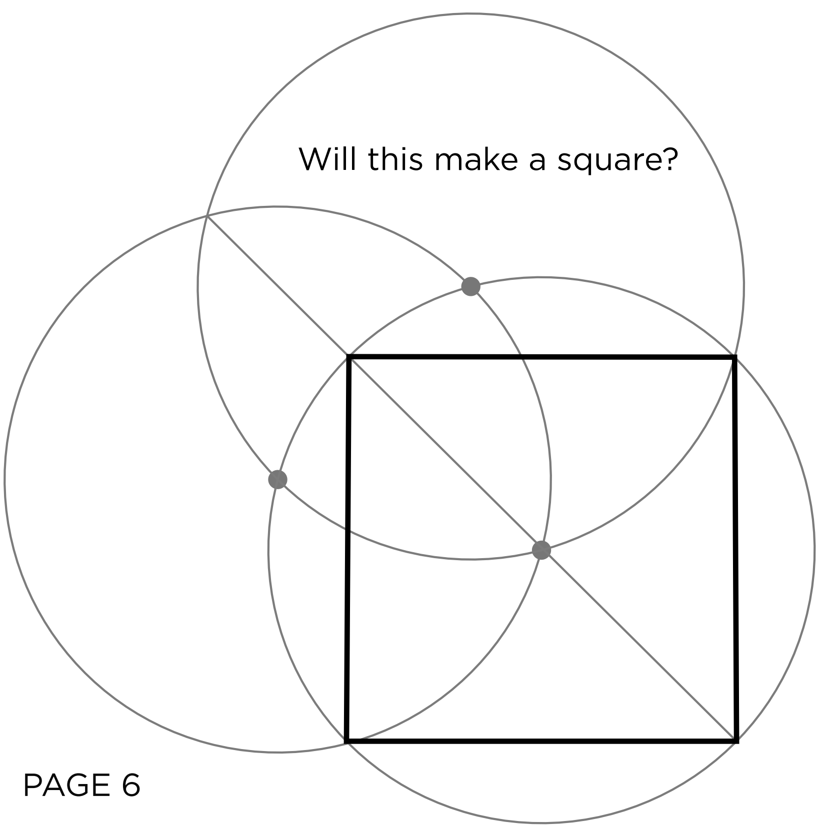


Will this make a square?

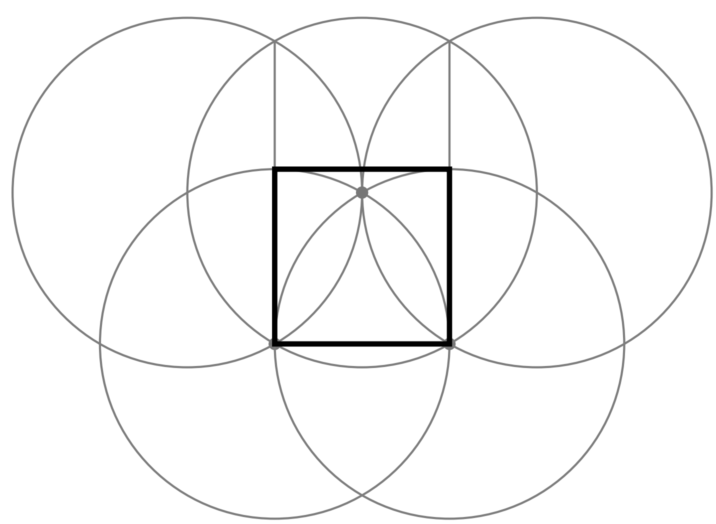


Can you prove it?

Will this make a square?



How about this?



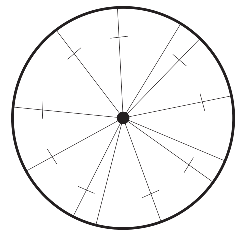
(Olin O-S, 2014)

The Rules of the Game

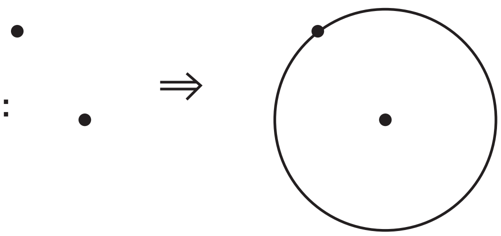
(Most of them at any rate)

Euclid's genius was to spell out the rules of the game. How can we prove, for sure, that a claim must be true? We start with some basic definitions and postulates, which we assume must hold. From these, we build up more and more true facts, theorems, which themselves become building blocks for even more facts!

Definition: All the points on a circle are the same distance from its center.



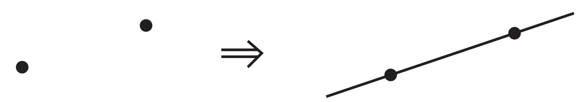
Given:



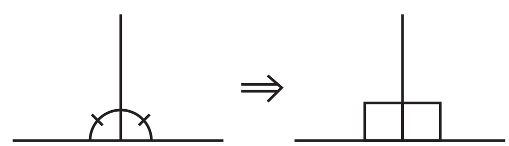
Postulate: Given any two points, you can draw a circle centered at the first point, passing through the other point. (You can do this with your compass!)

Postulate: You can draw a line through any two given points, and you can extend a line as much as you like. (You can do this with your ruler.)

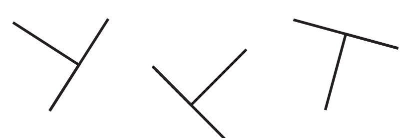
Given:



Definition: A right angle is defined in a funny way: if a line hits another line, making two angles that are equal to each other, these are, by definition, right angles.



Postulate: You can assume all right angles are equal to each other.



We won't need Euclid's Fifth Postulate until Proposition 29, but let's spell it out now!
Postulate: If two lines meet a third, and the interior angles sum to less than two right angles, the two lines must meet somewhere.

(We didn't have room to show where they meet!)

Let's Prove Some Stuff!

Our goal is to build up theorems using the postulates and earlier theorems as building blocks!

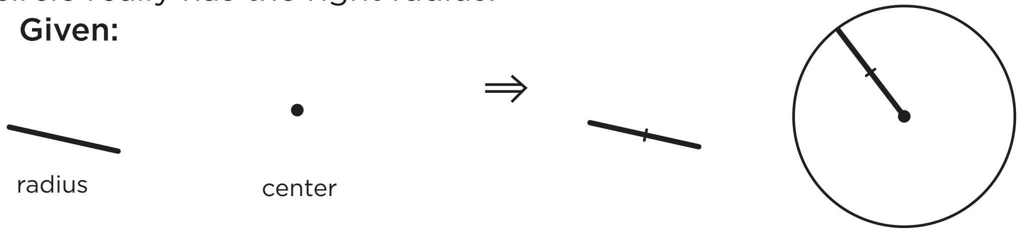
We've already shown how to construct an equilateral triangle and prove what we constructed really is an equilateral triangle like we said it is!

Euclid's Proposition 1.1 Given a segment, construct an equilateral triangle upon it.

Prove this is really an equilateral triangle!

1.2 Given a center and a radius, construct a circle with that center and radius using a floppy compass.

Prove your circle really has the right radius!



1.3 Given a longer line segment and a shorter one, cut the length of the shorter one off of the longer one.

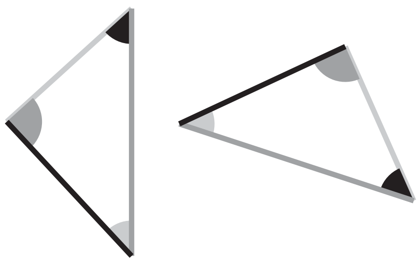
Prove this works!

1.4 (SAS) Suppose two triangles have a pair of matching sides, and a matching angle between them. Then the triangles are equal (all sides and angles match).

Do you believe this? What is the proof?

Before we go on, we should ask ourselves what is likely to be true?

Q: Two triangles are equal if the sides each match and all the angles each match. We could write this as saying ASASAS (angle, side, angle, side, angle, side) all match on the two triangles.



But what if less information is given?

If all the sides match (SSS), do the angles have to match too? See if you can draw two triangles that have all the same lengths of side, but somehow the angles are different.

Try it!

If two sides and one of the angles match, must the other side and the other two angles match too?

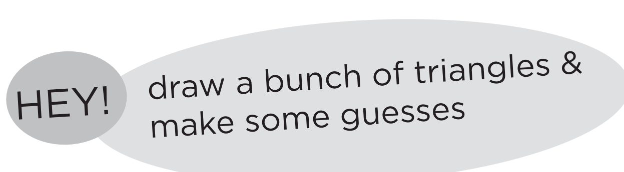
(Does it matter if the matching angle is between the two matching sides — SAS — or next to one of them — SSA?)

What if all the angles match — do the sides need to match also (AAA)?

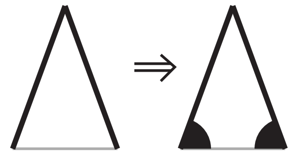
What if two angles match and one of the sides?
Does it matter if the matching side is between the two matching angles — ASA — or next to one of them — AAS?

In other words, which of these are probably theorems?

SSS SAS SSA AAA ASA AAS

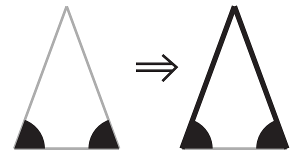


Here are the next few: Can you prove these?

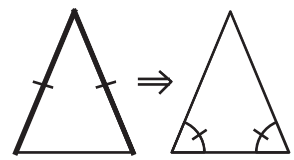


I.5 If a triangle has two sides equal, then the opposite angles are equal.

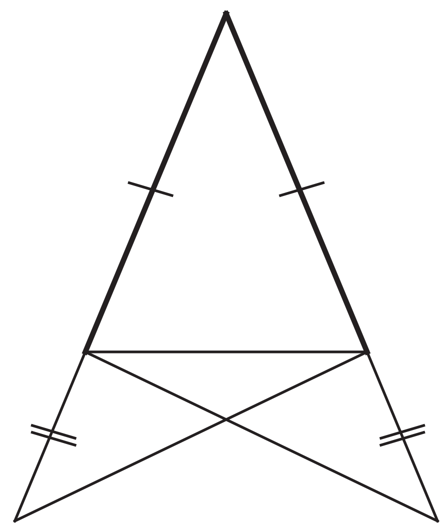
1.6 Conversely if a triangle has two angles equal, then the opposite sides are equal.



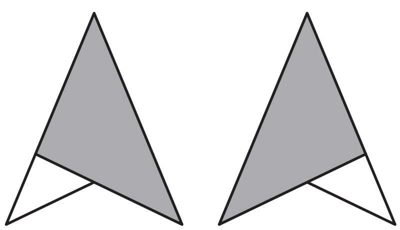
Let’s prove Prop 1.5 that if two sides of a triangle are equal, the opposite angles are equal as well. All we need is to use SAS, over and over!



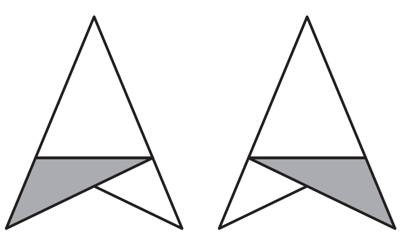
Proof: Given a triangle with two sides equal, extend the sides an equal amount and connect as shown



Q: Then why must these two triangles also be equal? What angles and segments must be equal since these triangles are? Mark this on the diagram!

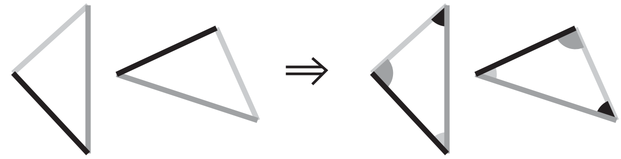


Q: Then why must these two triangles also be equal? What angles and segments must be equal since these triangles are?



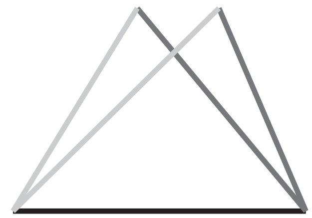
Finally, knowing what we know about which angles are equal so far, how do we finish the proof of the theorem?

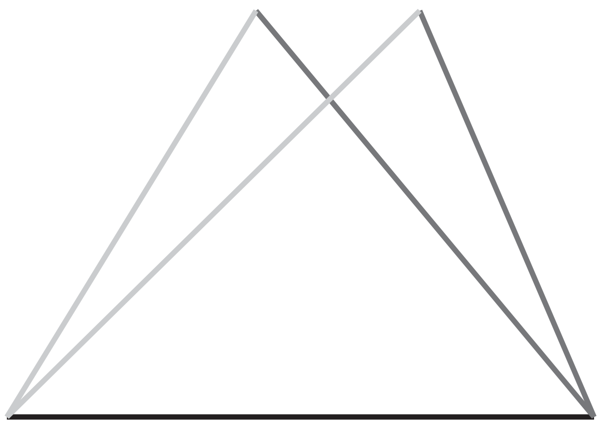
1.7 & 1.8 (SSS) If all three sides match between a pair of triangles, all the angles match as well.



We can line the two triangles up; if SSS isn't true, we'd have a picture like this one at right. So all we really have to do is to prove this picture is impossible.

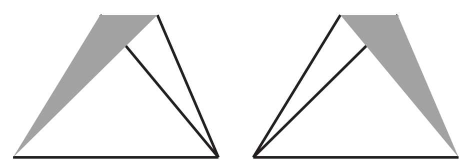
(Edges of the same color are supposedly the same length.)



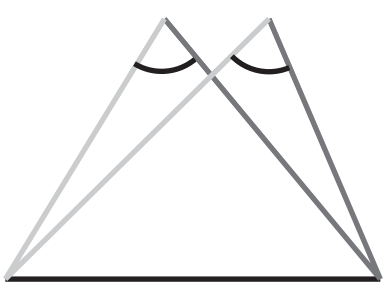


Hint: Connect the top points. We have two "isosceles" triangles.

What do we know about their angles?



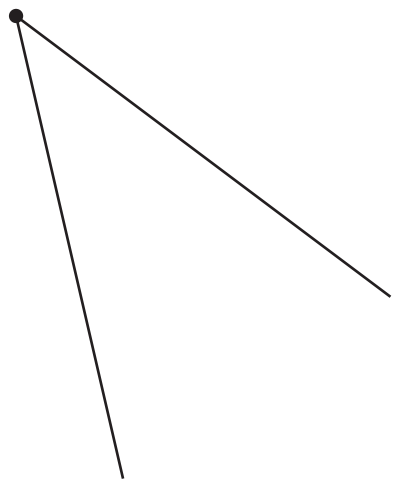
Which of these angles must be larger if the diagram is legitimate?



Conclusion: Ooops! The diagram couldn't be legitimate. This can't happen. The triangles must have been equal to begin with.

Let’s assume that SSS is true.
(We will prove this soon, but let’s see what it can give us now.)
So we can use 1.4 (SAS), 1.5 (isosceles triangles have equal angles on the base) and SSS (1.8)

1.9 Bisect an angle.



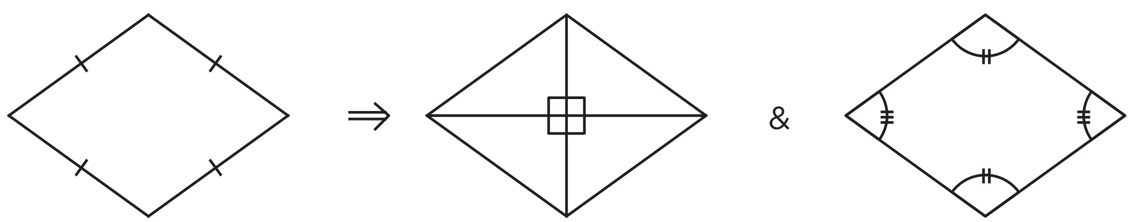
Prove that this works:

1.10 Bisect a segment.

Prove that this works:

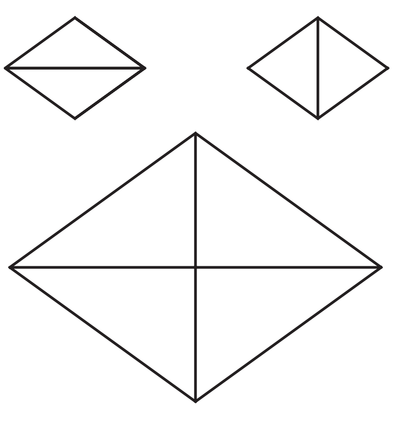
1.11 Construct a perpendicular to a given segment at a given point
— and prove your construction always works!

Theorem (not in Euclid):
Opposite angles on a rhombus are equal,
and the diagonals bisect each other and meet at right angles.

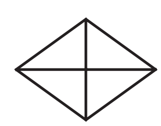


Proof: Draw the diagonals.
As we proceed, mark what angles we know must be equal.

Why are these triangles isosceles?
What does that tell us about their angles?
Why does that tell us they are equal?
Finally, how can we conclude the top and bottom angles in the rhombus must be equal?



Using the same argument, show the left and right angles in the rhombus must be equal.

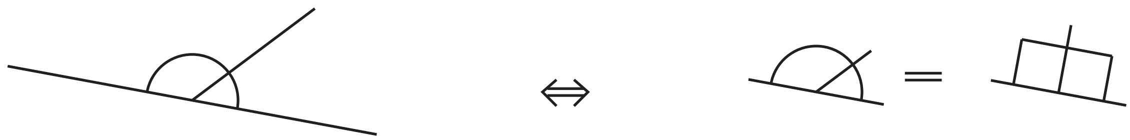


And why must the diagonals bisect each other?

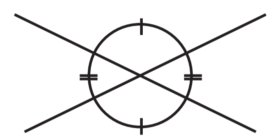
Finally, why are these smaller triangles all equal, and how does that tell us that the diagonals must bisect each other and meet at right angles?

More Euclid!

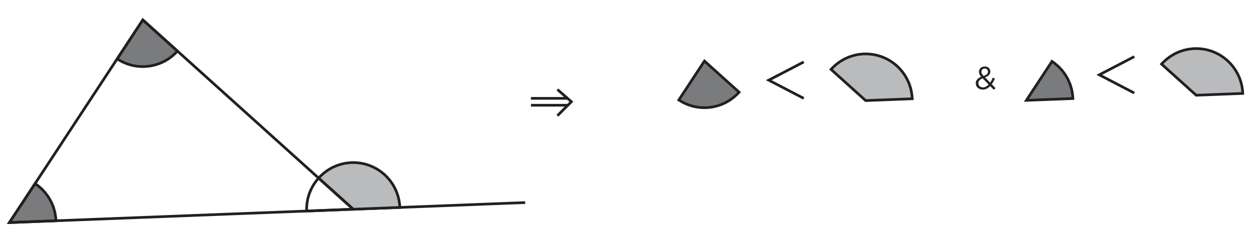
I.13, I.14: The sum of two supplementary angles equals two right angles (not a big deal: construct a perpendicular using 1.11, then arithmetic). Conversely, if two adjacent angles sum to two right angles, they lie on a straight line.



I.15: Vertical angles are equal.
(The proof is arithmetic, using 1.13.)



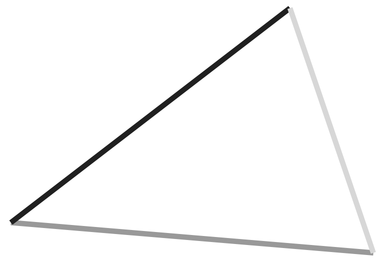
I.16 In a triangle, consider two angles opposite to a third; then either of the two angles is less than the supplement of the third. (Let's see a picture!) Even though this theorem doesn't say very much, its proof isn't very long, and the theorem is an important stepping stone.



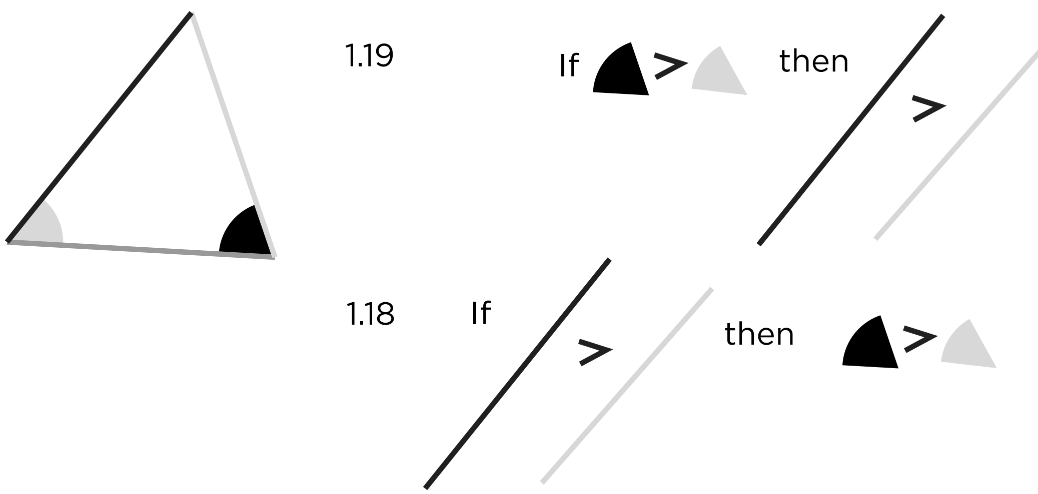
This is a surprisingly powerful theorem and has a very clever proof!

The Triangle Inequality, and how to construct a triangle given its sides:

Euclid 1.20 is the famous Triangle Inequality:
If the sides of a triangle have lengths A, B, and C
A+B must be less than C,
B+C must be less than A,
C+A must be less than B.



In order to pull this off, Euclid uses 1.19, which uses 1.18, which uses 1.16.



And Euclid 1.22 tells us how to construct a triangle with three specified side lengths (if they satisfy the triangle inequality). For practice, construct a triangle with these three side lengths:

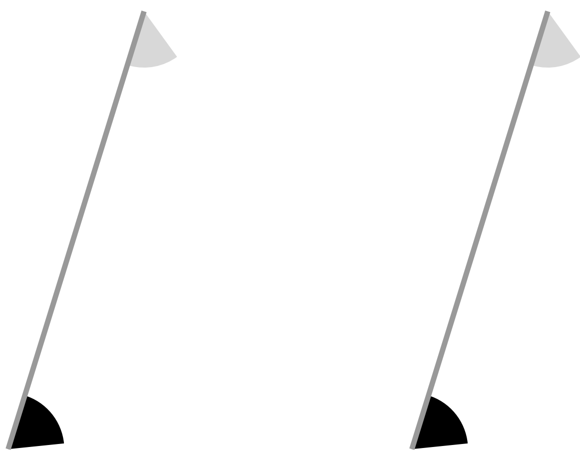
We've seen

1.26 ASA, AAS!

SAS (Proposition 1.4)

SSS (Propositions 1.7 & 1.8)

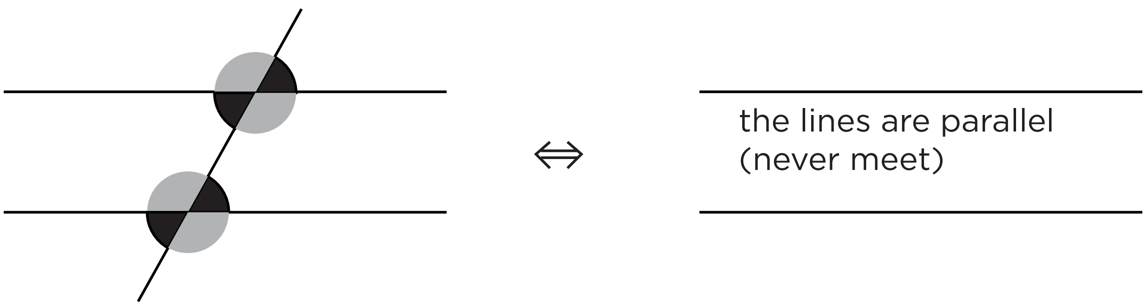
AAA is just false. SSA is also false, but a little surprising how.
That just leaves ASA and AAS: Proposition 1.26.

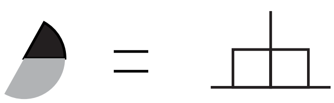


A curious note: Euclid groups ASA and AAS together and waits until Proposition 26 to get around to proving these are true. But if you look closely, his proof of ASA could have been given any time after Proposition 4 — I wonder why he waited so long!

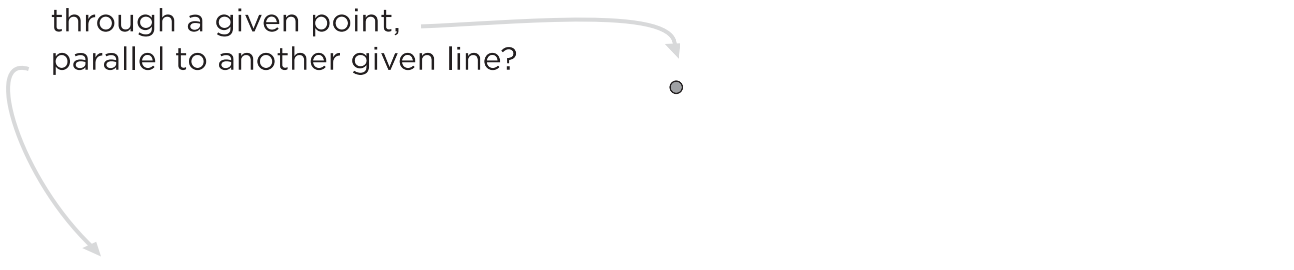
And then we're ready to begin talking about parallel lines! This is a big step, for it is the first time we'll be using the mysterious fifth postulate. All the theorems so far hold in any geometry that satisfies just the first four postulates: on the sphere, the plane, and the hyperbolic plane. But once we use the fifth postulate, our results hold only in the plane.

Theorems 27, 28, 29:





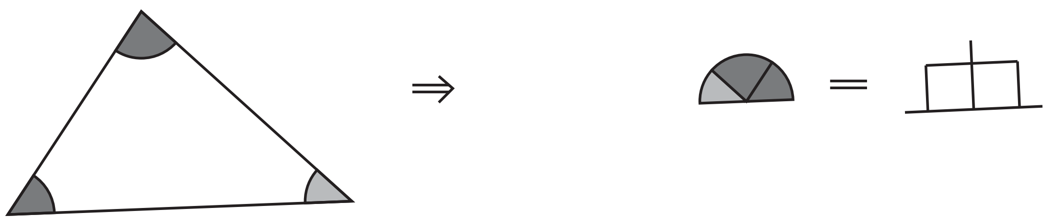
Theorem 31: How do you construct a line,
through a given point,
parallel to another given line?



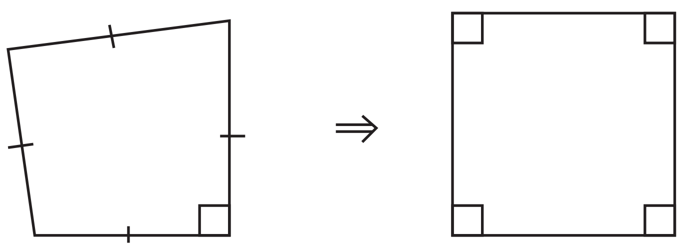
More Theorems that Require the Fifth Postulate

From these we'll be able to prove some facts that only hold in the Euclidean plane, where the fifth postulate holds. (They don't hold in non-Euclidean geometry!)

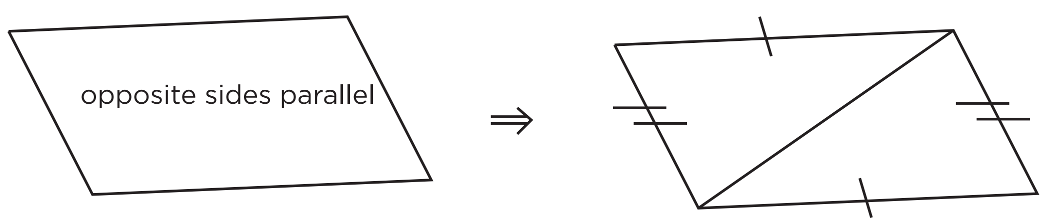
1.32 The sum of interior angles of a triangle equals two right triangles.



If all sides of a quadrilateral are equal and one interior angle is a right angle, then all angles are right angles and the quadrilateral is a square. (We needed this on the first day, when we claimed we had really constructed a square!)



I.33 Opposite sides of a parallelogram are equal, and the diagonal of a parallelogram divides it into two equal parts.



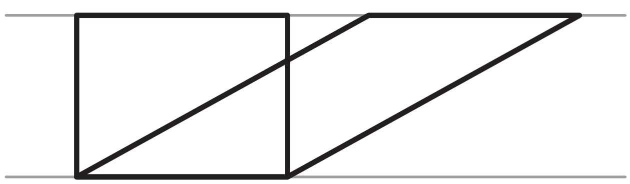
I.47 The Pythagorean Theorem!!

A Very Cool Proof

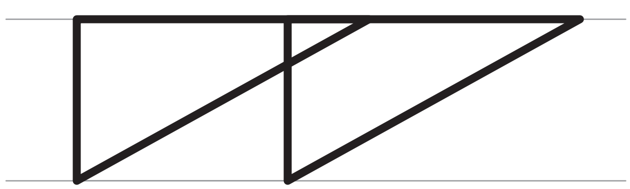
Once we prove that opposite sides of a parallelogram are equal, we can prove:

Parallelograms with equal bases, between the same parallel lines, have the same area.

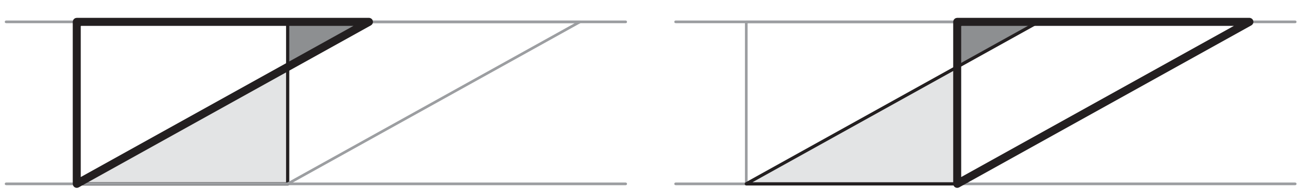
A hint of the proof. We begin with:



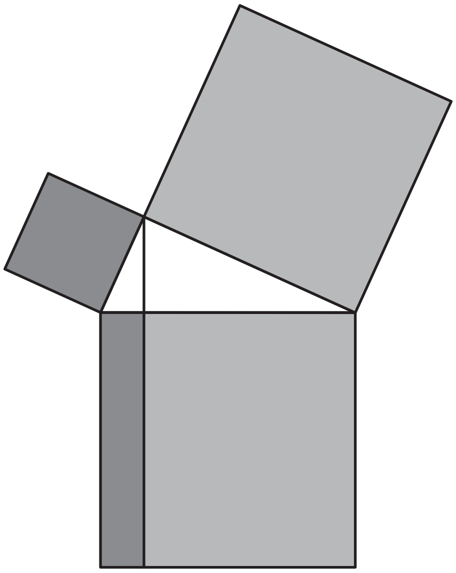
Why are these two large triangles equal?



What do we need to add and subtract from the triangles to get the area of each parallelogram?



Euclid's Proof of the Pythogorean Theorem (1.47)

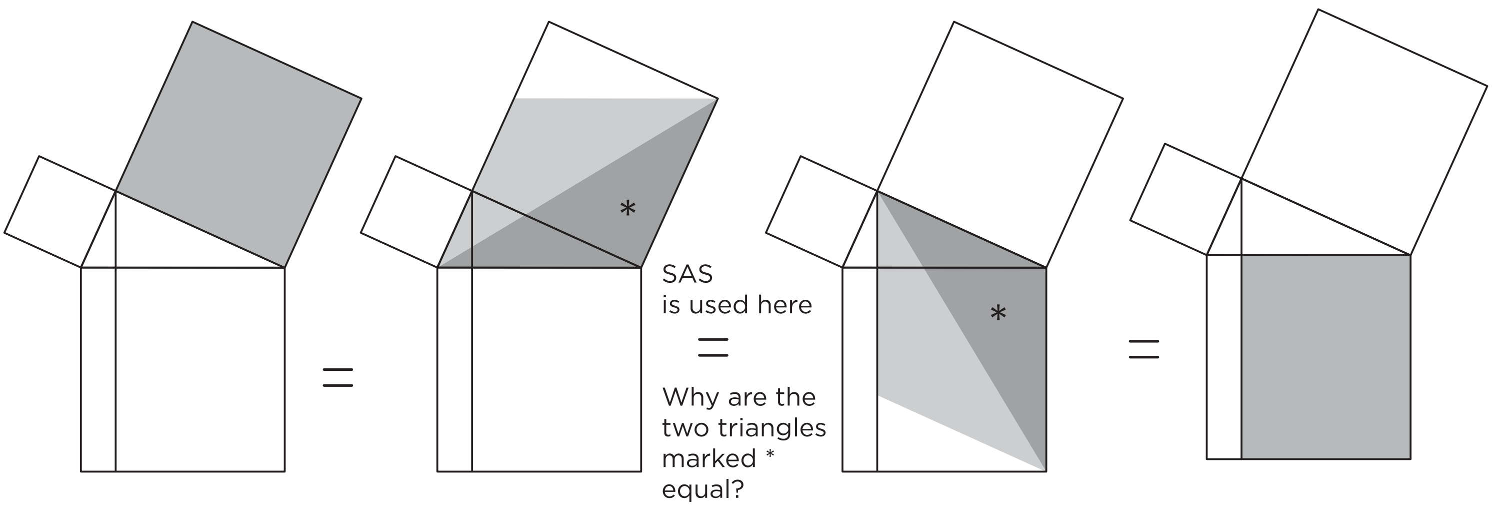


We will show that the shaded regions have the same area, using three theorems:

- A diagonal bisects a parallelogram into two equal areas.
- Parallelograms with the same base and between the same parallels have the same area.
- SAS.

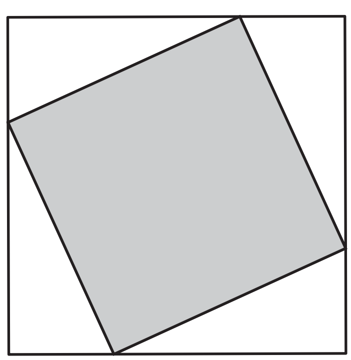
We'll give the argument for just one of the areas: the argument for the other is exactly the same.

Why is each step valid?

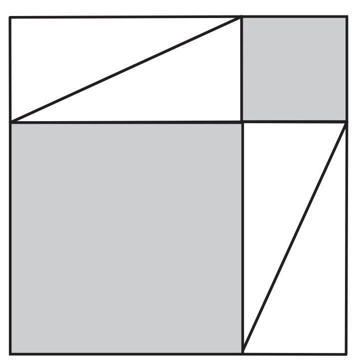


Here is another proof, known in China 2000 years ago; the Hindu mathematician Bhaskara simply wrote:

BEHOLD!



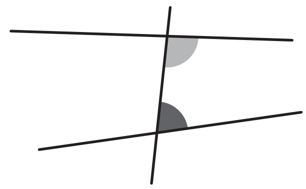
he was referring to:



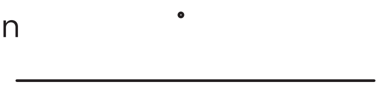
Which of Euclid's propositions are needed for this proof to be complete?

Different Possible "Parallel Postulates"

Any of these could have been taken as ***the*** Parallel Postulate and we would have had the same geometry.

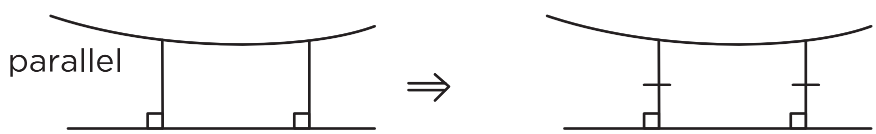
Euclid's Parallel Postulate: Given  with 

then the lines must meet.
(Or conversely, if the lines are parallel, the interior angles are at least two right angles)

Playfair's parallel postulate: Given 

There is a unique line through the given point, parallel to the given line.

Tom's Postulate: If two lines are parallel, then any two segments between them, perpendicular to one of the lines, have the same length.



Any of these can be taken as an axiom and the others proven as theorems. For example:

Taking Euclid's postulate, prove Playfair's and Tom's.

Taking Playfair's, prove Euclid's and Tom's.

Taking Tom's, prove Euclid's and Playfair's.

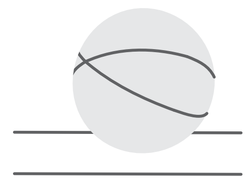
In any of these, you may use all the theorems that rest on the first four postulates, namely Propositions 1–28.

A quick word about Non-Euclidean geometry.

There is more than one kind of geometry in which the first four postulates hold!

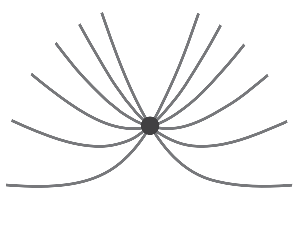
The most important examples are elliptic geometry, which is the geometry of the sphere, and hyperbolic geometry.

Elliptic Geometry



On a sphere, we can define lines to be great circles — circles that cut the sphere in half. The first four postulates work just as well, and all the theorems that depend upon them. We still have SAS, SSS, ASA, facts about isosceles triangles, and all the constructions we've seen so far.

But there are no parallel lines — every pair of great circles intersect — and we have some surprising theorems, such as: The sum of angles in a triangle is greater than two right angles.



Hyperbolic Geometry:

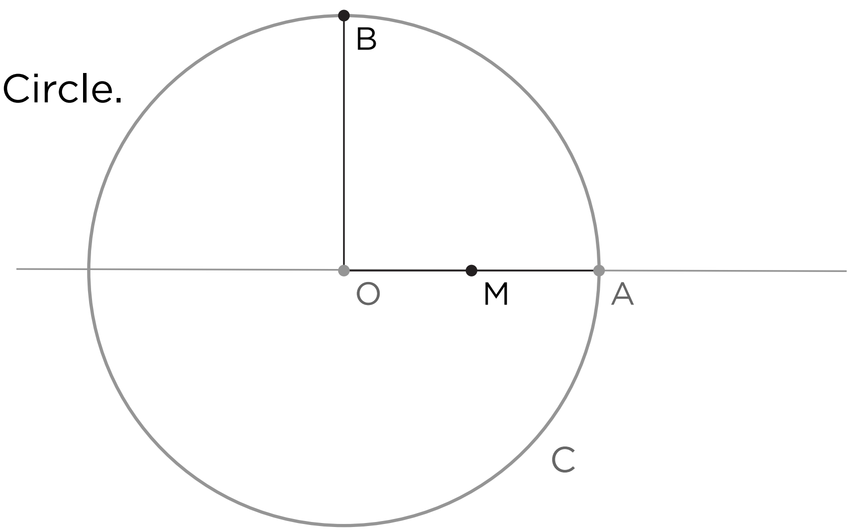
We'll need to take some time to explain hyperbolic geometry, but in essence it is the geometry of surfaces with a lot more room than in the plane — the surface of a wrinkly lettuce leaf is an example. In hyperbolic geometry, there are a great many lines, parallel to a given one, through a given point! And we have theorems like this one: The sum of angles in a triangle is less than two right angles!

One thing that holds in both elliptic and hyperbolic geometry, but not in the plane, is AAA — two triangles with the same angles have to be congruent!!

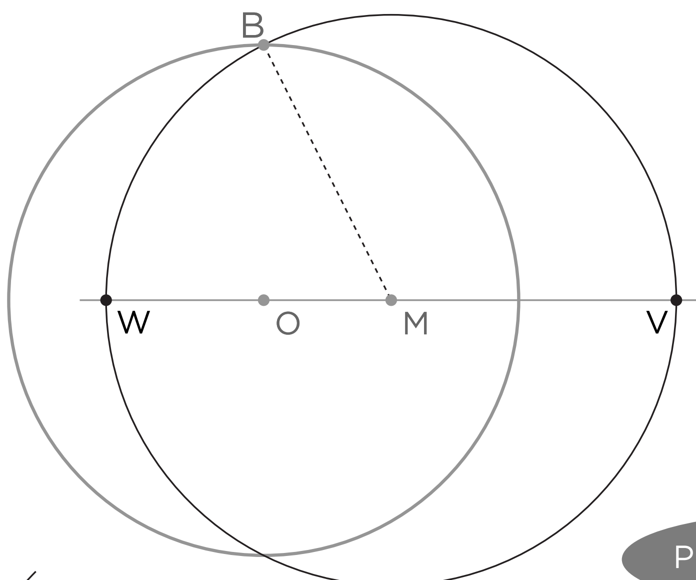
For example, on the sphere can you find a triangle that has three right angles? Can tell why any two triangles like this are congruent?

Euclid's Construction of a Regular Pentagon, Given a Circumscribing Circle.

Begin with a circle C with center O. Draw any line OA passing through O. Find the midpoint M of segment OA. Draw a perpendicular radius OB.

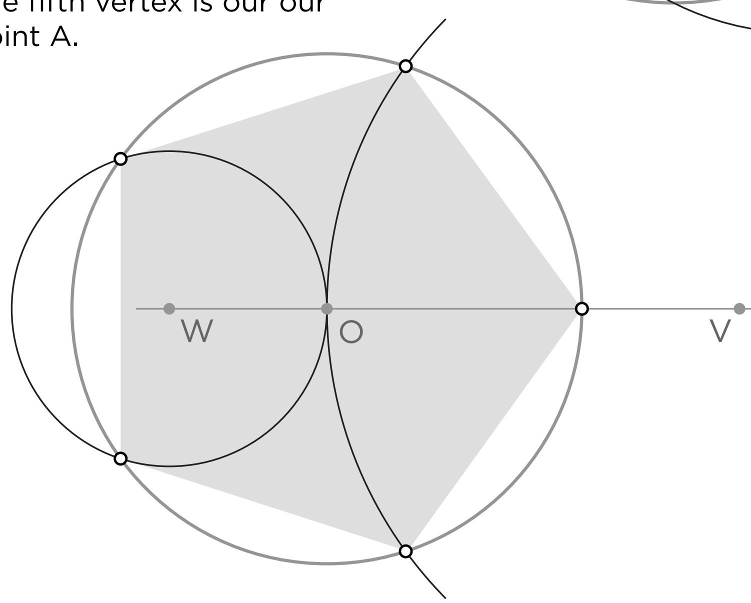


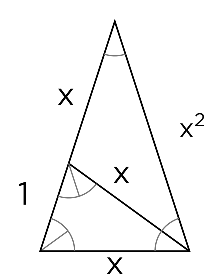
Draw the circle with center M, passing through B, and let W and V be the intersections of this circle with the line OA.



Once we have V and W, construct the circles with centers V and W passing through O. The intersections of these circles with C are four of the five vertices of the desired pentagon. The fifth vertex is our our point A.

Prove this works!



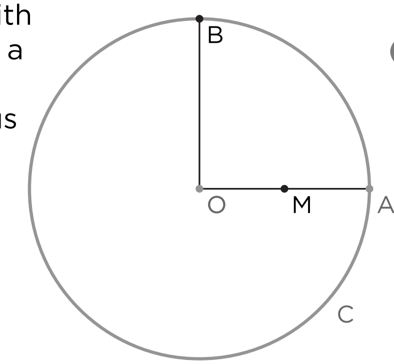


A 36-72-72° isosceles triangle has both base angles twice the top. Subdivide into two isosceles triangles as shown. Taking 1 and x as shown, by similarity we find that the sides $x^2 = 1 + x$.

Solve for x. Can you see how this quantity is constructed in our construction?

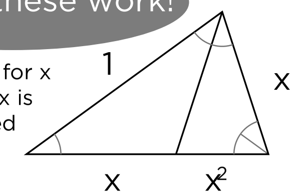
Four Constructions of a Regular Pentagon, Given a Circumscribing Circle.

All four constructions begin with a circle C with center O. Draw a radius OA and find its midpoint M. Draw a perpendicular radius OB.

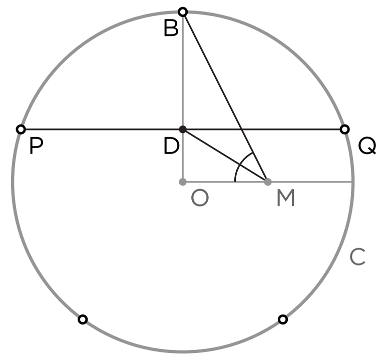


Prove these work!

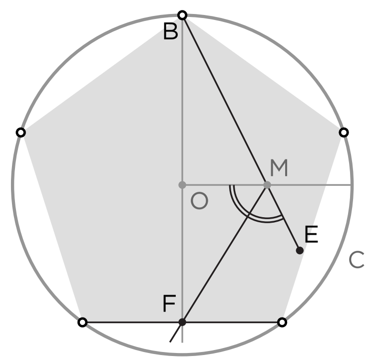
hint: solve for x
Show this x is constructed



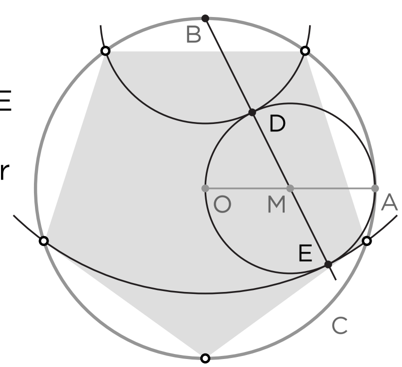
❶ Richmond's construction: construct the bisector of angle OMB and let D be the intersection of OB with this bisector. Construct the line parallel to OM, passing through D. Let P and Q be the intersections of the circle C with this parallel. P, Q, and B are three of the vertices of the desired pentagon.



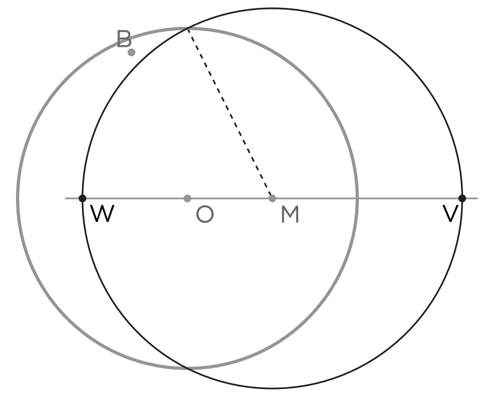
Two vertices are enough to find all of the others, but it's nice to see how Richmond's construction can be continued: Extend the ray from B through M to some point E. As before, construct the bisector of angle OME, and let F be the intersection of this bisector with the ray BO. Constuct the parallel to OM passing through F — the other two vertices of the pentagon lie at the intersection of this parallel with C



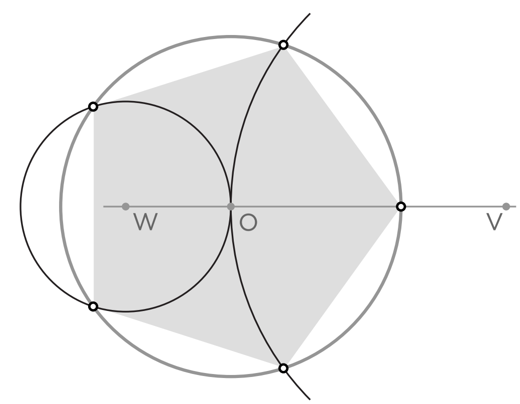
❷ Draw the ray from B passing through M, and the circle with center M passing through O and A. Let D and E be the intersections of this ray with this circle. Draw the circle with center B passing through D, and the circle with center B passing through E. The intersections of these circles with the original circle C are four of the five vertices of the desired pentagon. The fifth vertex R is opposite B on the circumference of C.



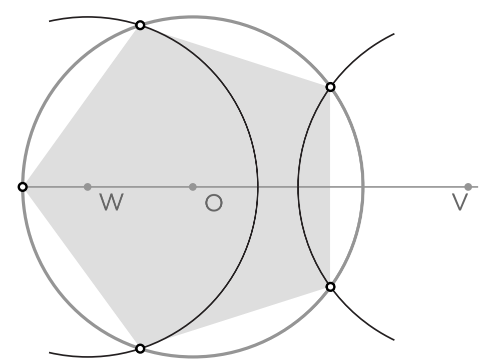
For the third and fourth constructions, draw the circle with center M, passing through B, and let W and V be the intersections of this circle with the line OA.



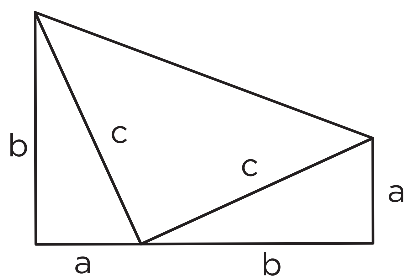
❸ Euclid's construction: Once we have V and W, construct the circles with centers V and W passing through O. The intersections of these circles with C are four of the five vertices of the desired pentagon. The fifth vertex is our our point A.



❹ Or we can use "Carlyle circles": construct the circles with the same radius as the circle C, but centered at V and W. The intersections of these circles with C are four of the five vertices of the desired pentagon. The fifth is the point opposite A on the circumference of C.



Here's a proof of the Pythagorean Theorem, due to President James A. Garfield (March 1881-Sept 1881):

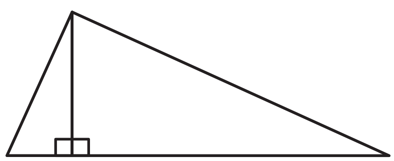


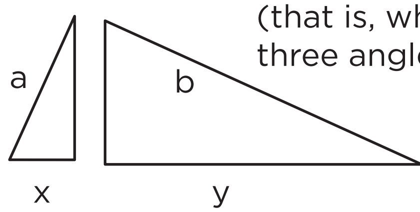
The total area of the trapezoid is ½ (a+b) (a+b).
The area of the triangles is ½ ab + ½ ab + ½ c^2
Can you use this to show $a^2 + b^2 = c^2$?

A full proof requires knowing the area of a trapezoid, which is not hard to prove using Euclid Book 1, the area of a triangle (which Euclid more or less works out in Book 1), and some algebra, which Euclid works out in Book 2.

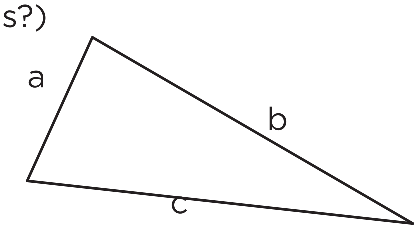
Later in the *Elements*, Euclid works out how to deal with ratios (Book 5) and that corresponding parts of similar triangles are in the same proportion (Book 6). He can then give the following proof (Book 6, Proposition 31).

Draw a perpendicular from the hypotenuse to the opposite vertex:





Why are all three triangles similar? (that is, why do they have the same three angles?)



It's easier for us to follow if we use our familiar notation. Explain each step:

$a / x = c / a$
Therefore $a^2 / c = x$
Therefore $a^2 / c^2 = x / c$

Meanwhile:

$b / x = c / b$
Therefore $b^2 / c = y$
Therefore $b^2 / c^2 = y / c$

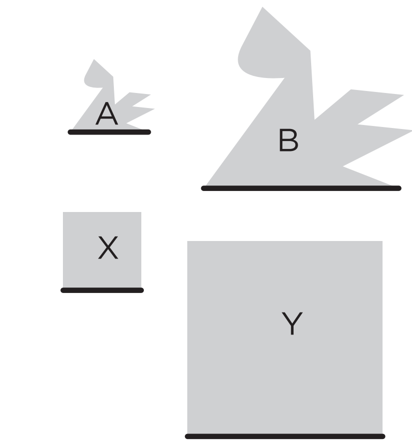
Now consider the sum $x/c + y/c$. On the one hand $x/c + y/c = (x+y)/c$ = _______

On the other hand substituting in, we have $x/c + y/c$ = ________

Finish the proof.

There are hundreds! See:
http://www.cut-the-knot.org/pythagoras/index.shtml

This is a stunning variation on the proof in Book 4, Proposition 31.
It relies on the following fact, proven in Euclid Book 4:



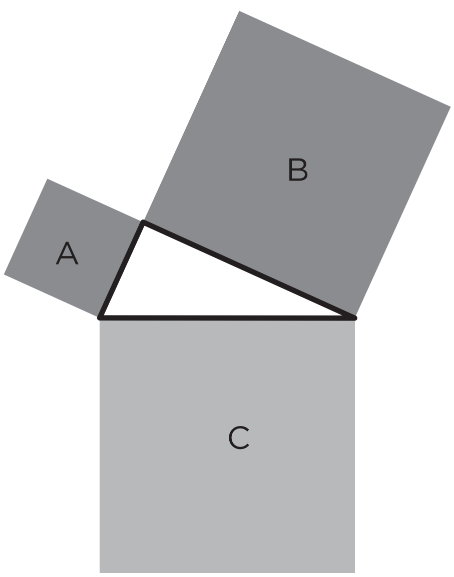
Suppose figures A and X are on the equal bases, and figures B and Y are also on equal bases;
further suppose A is similar to B and X is similar to Y.

Then (area of A) / (area of B) = (area of X) / (area of Y)

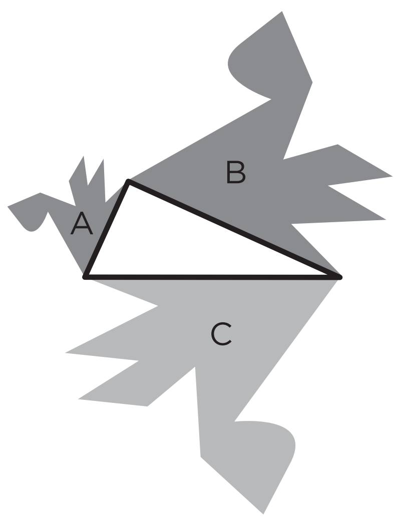
That is, the areas are in the same proportion.

What this means is that if I want to show that the areas of the two squares add up to the area of the third, it suffices to show this for any figures that have the same bases.

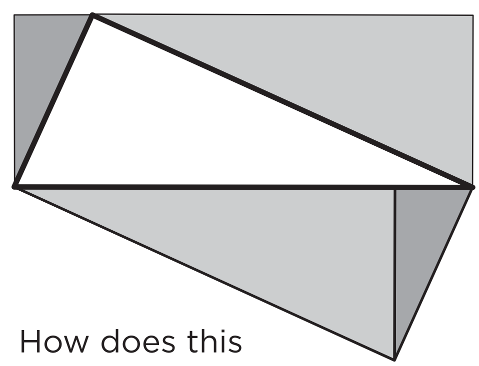
SO: we might as well use a clever shape for the proof!



areas A+B= C $\Leftrightarrow$



areas A+B= C



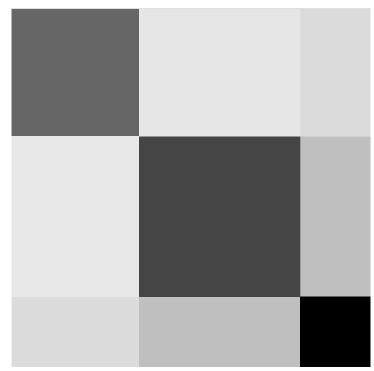
How does this complete the proof?

An Introduction to Book 2

In Euclid’s time, algebra had not yet been invented. Euclid and others did need algebraic facts. Book 2 of Euclid’s Elements catalogues many clever geometric work-arounds for facts more easily handled algebraically.
Today these remain very nice “proofs by picture.”

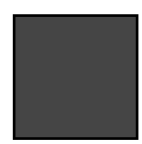
For example:

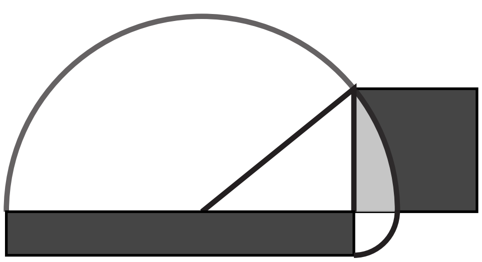
Theorem: $(a+b)^2 = a^2 + 2ab + b^2$ Proof:



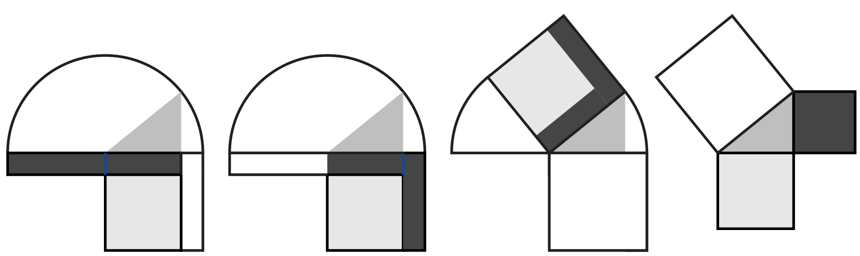
Theorem: $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2ab$
Proof:

Euclid gives a clever construction of the harmonic mean:
Given a and b, construct $\sqrt{ab}$

Given  produce a square with the same area: 

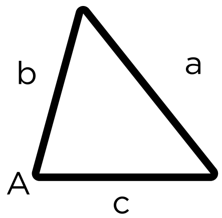


Proof:

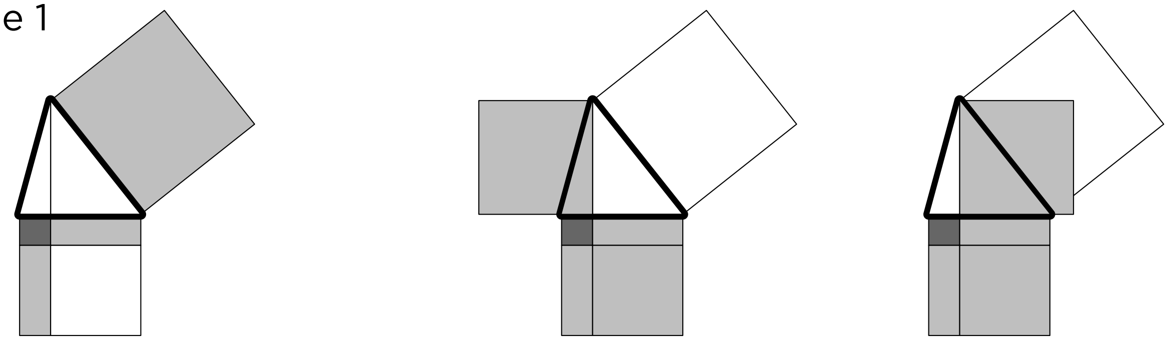


Euclid’s proof of The Law of Cosines (II.13)

$a^2 = b^2 + c^2 - 2\,bc \cos A$

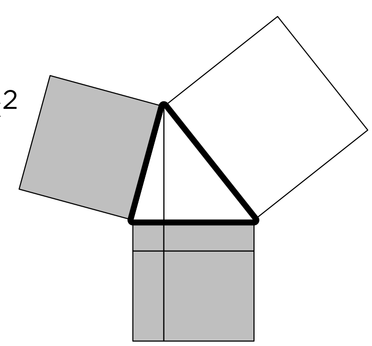


Case 1



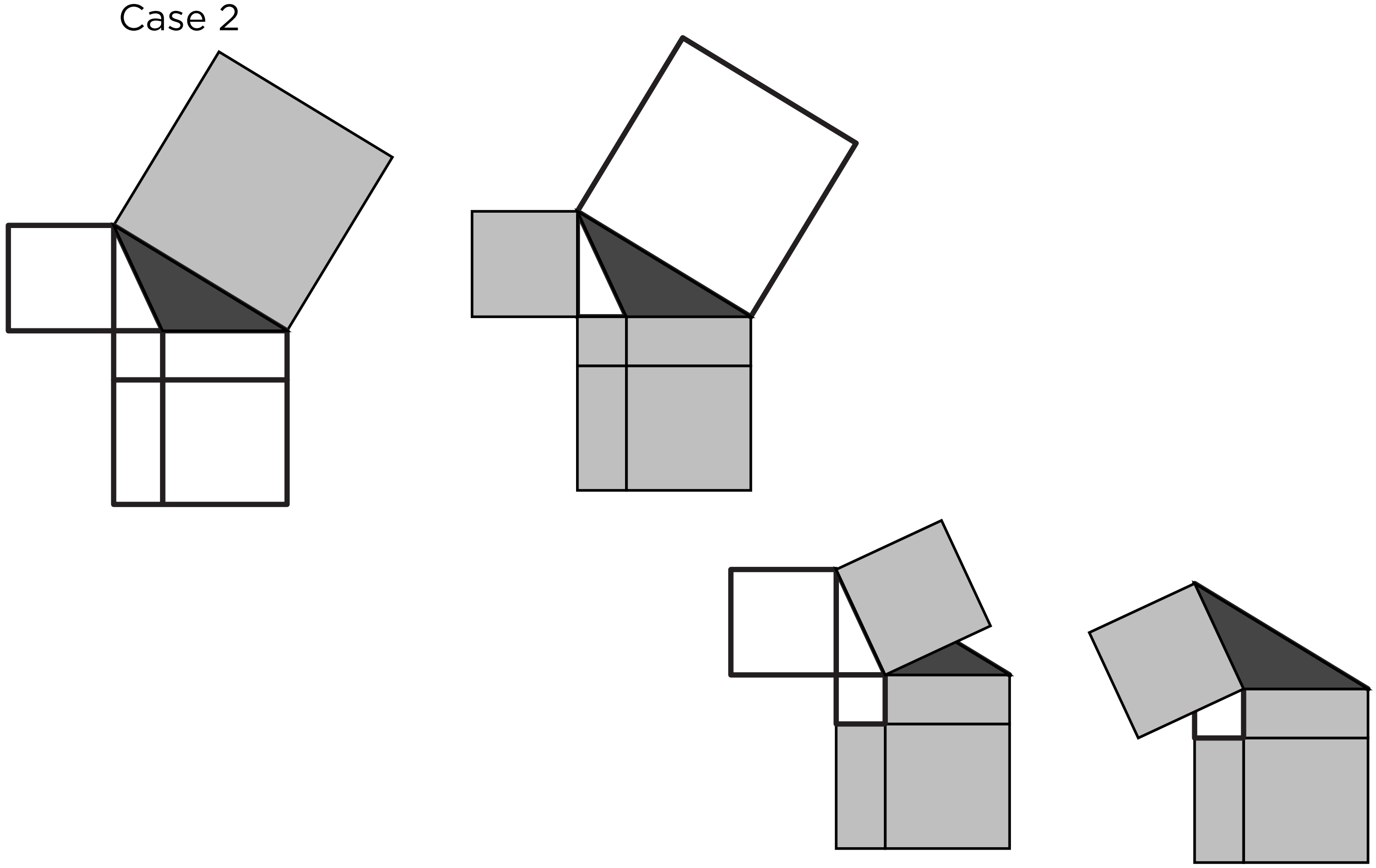
$a^2 + 2\,bc \cos A =$

$= b^2 + c^2$



(and apply the
Pythogorean theorem)

Case 2

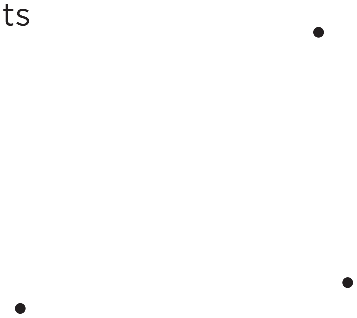


Some important facts from Book 3 of *Euclid's Elements*:

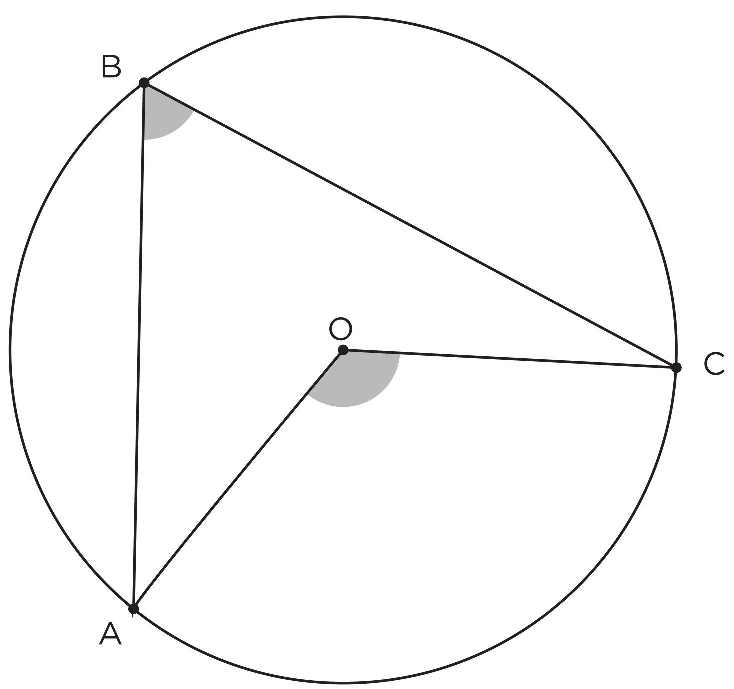
Can you supply the proofs?

Construct the circle through three given points

(These require the Parallel Postulate!)

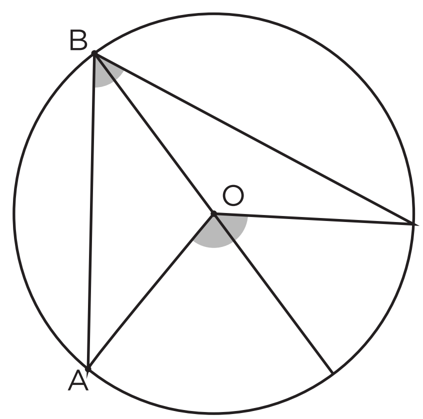


Thereom 3.20 is remarkably useful:
In any diagram like this one, angle ABC is half of angle AOC. What is the proof?
(This mainly uses Proposition 1.6, that base angles of an isosceles triangle are equal.)



Hint: Split into two parts using the diameter BO. Look at each part separately.

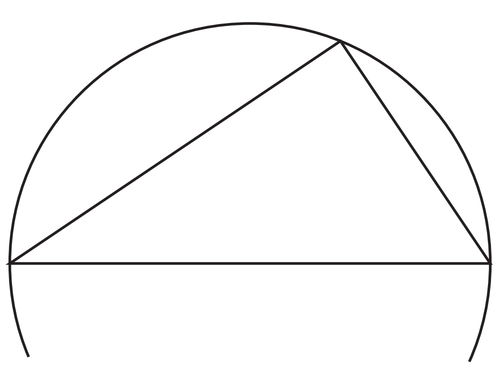
Do you see a lot of isosceles triangles?



(How does this proof use the parallel postulate?)

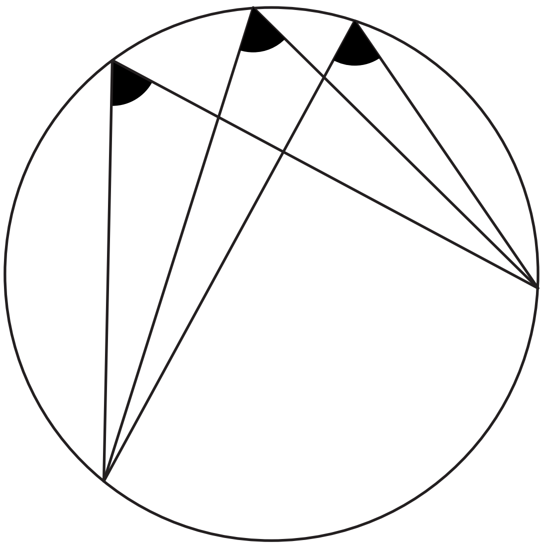
This theorem has some remarkable corollaries:

Corollary (Any triangle with vertices on the circle, and one side a diameter of the circle, must be a right triangle!

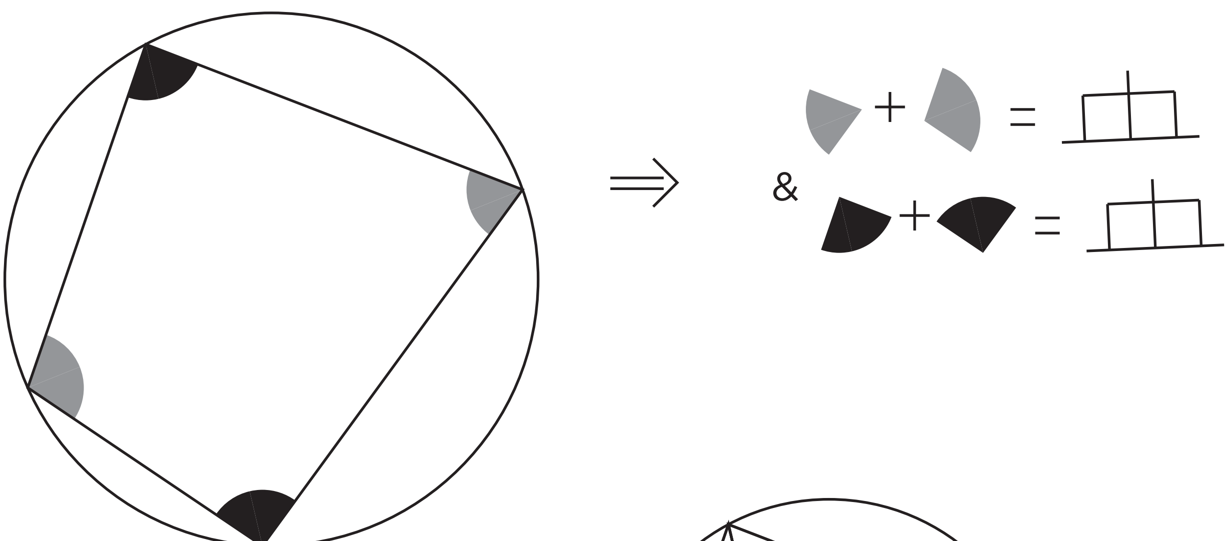


Corollary (3.21) Incredibly, if angles on a circle subtend the same arc, they must be equal.

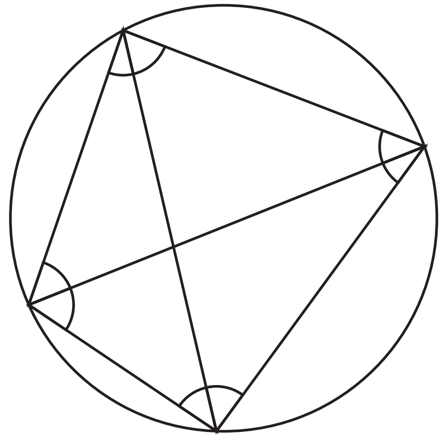
Can you supply the proofs?



Corollary (3.22) If all the vertices of a quadrilateral lie on a circle, then opposite angles must sum to two right angles.



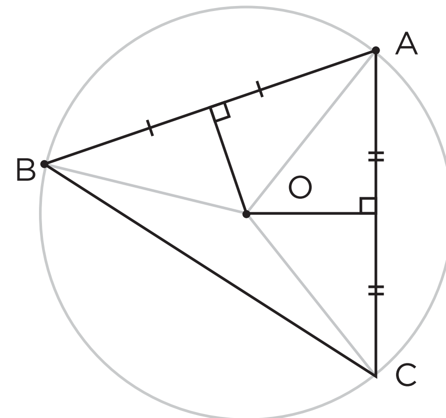
Hint: Use this diagram,
Theorem 3.20,
and the basic fact that
the sum of the internal angles
of a triangle equals two
right angles (1.32).



Some facts about circles and triangles.

We have already seen how to construct the circle through three given points:
(Euclid 5.5) Construct the perpendicular bisectors to two of the legs of the triangle. The point where they meet will be the center of the triangle.

But why does this work? Given points ABC, construct the perpendicular bisectors to AB and AC, and let them meet at point O. (Why must they meet at all?)



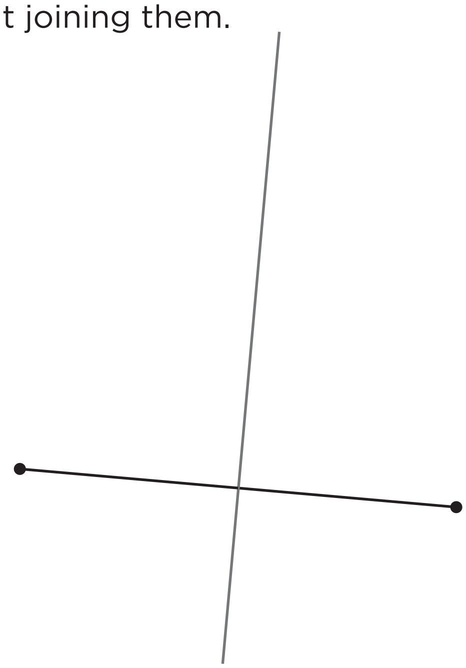
Then we must show that OA, OB, and OC are all of equal length. But that's just a few applications of the Hypotenuse-Leg Theorem (which follows from the Pythagorean Theorem).

Thus, the circle centered at O, passing through one of the vertices of the triangle also passes through all three.

Note that this implies, and is implied by:

Every circle through any two given points lies on the perpendicular bisector of the segment joining them.

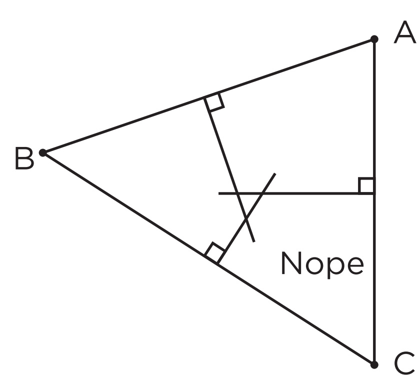
Try this:



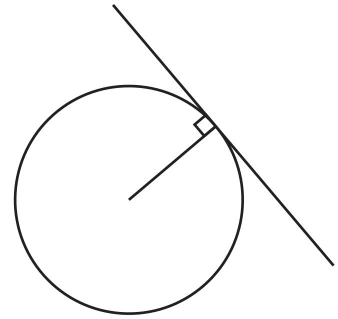
Draw a bunch of circles on the perpendicular bisector of this segment, passing through both ends.

I think this is a pretty cool picture!

The construction also implies an interesting thing: the perpendicular bisectors must all meet at a point (the center of this circle). Initially, it's not clear that the perpendicular bisectors couldn't look something like this instead:



Euclid 3.16 tells us that any line tangent to a circle at a given point must be perpendicular to the radius of the circle at that point. We'll use this for our next construction.

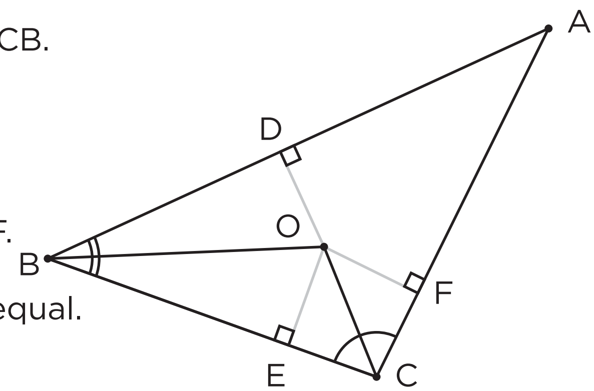


Let's construct the circle inscribed in a given triangle (4.4). This is really very pretty:

Construct the angle bisectors of angles ABC and ACB.

Let O be the point where these meet.

From O, construct the perpendiculars to each of the segments, meeting at points D, E, and F.

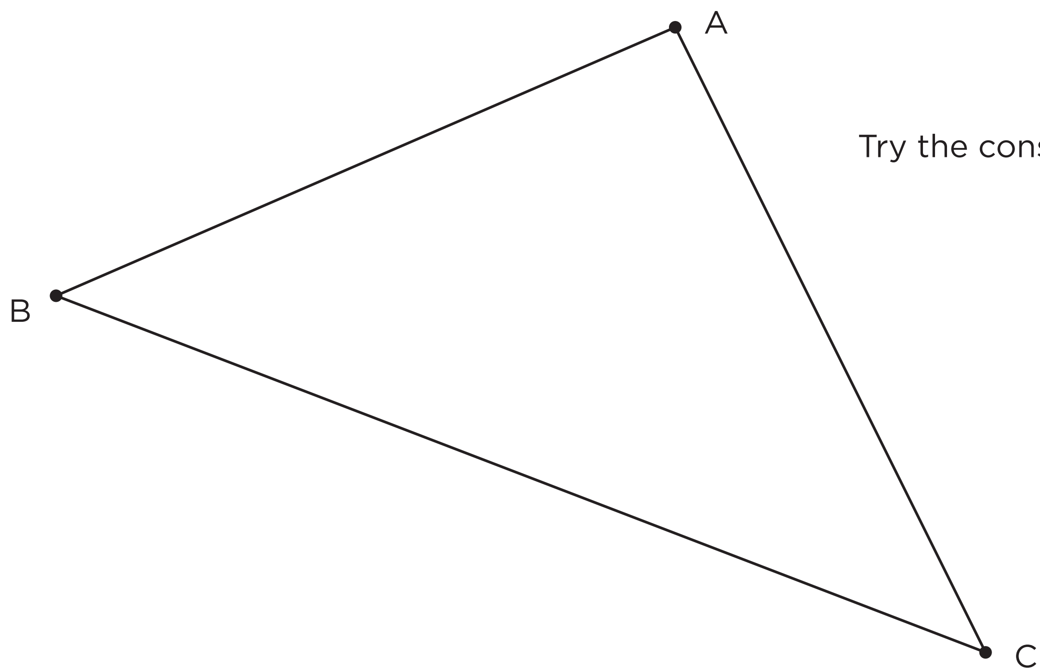


The claim is that segments OD, OE, and OF are all equal.

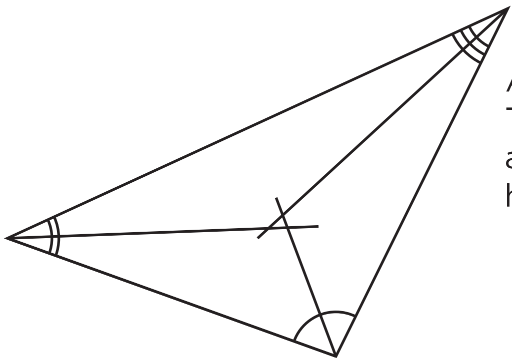
Why is this true?

So the circle centered at O, passing through one of the points D, E, or F, passes through all three of them.

Why must this circle be tangent to the sides of the triangle?



Try the construction yourself.

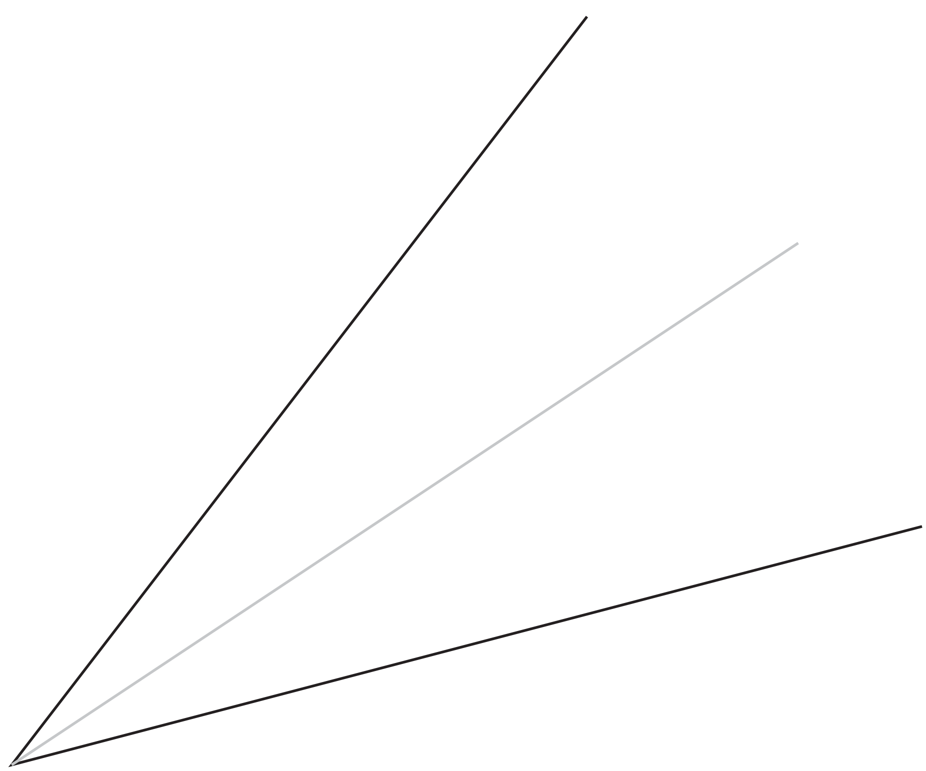


And we get another interesting observation: This implies that all the bisectors of the interior angles must meet at a single point. We can't have a picture like this one. (Why not?)

This also implies, and is implied by, the following observation:
If a circle is tangent to two lines that meet,
its center must lie on the angle bisector.

Once we have found the center of this circle, a perpendicular from the center to a side will meet that side in a point that lies on the circle, allowing us to complete the construction.

Try it!



Another theorem, which is quite nice:

In a triangle, we've seen that the perpendicular bisectors of the sides must meet at a point.

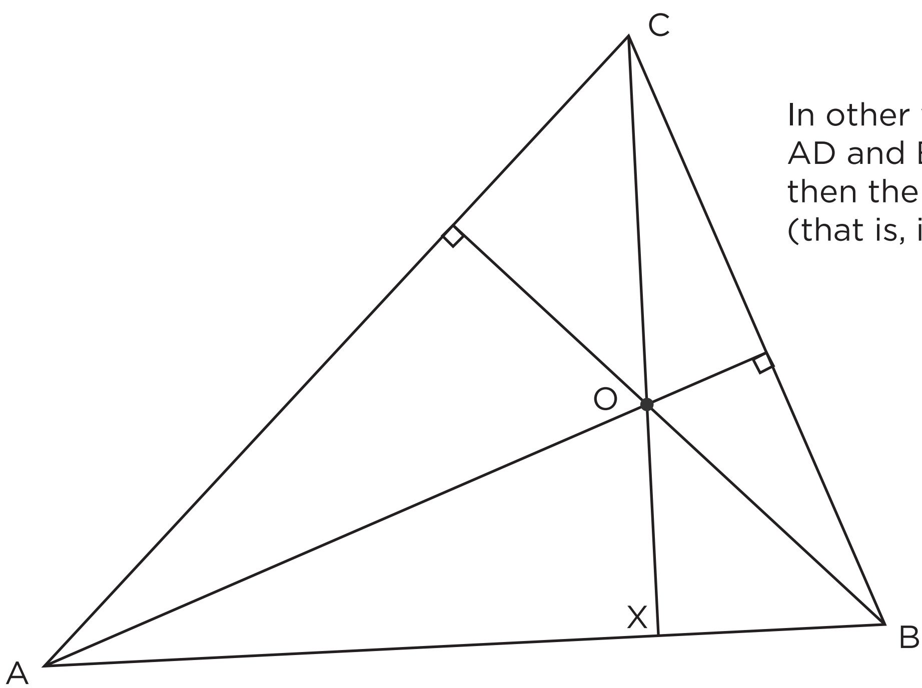
We've seen that the bisectors of the angles must meet at a point.

An altitude of a triangle is a segment perpendicular to a side, through the opposite vertex.

Construct the altitudes in this triangle.

Amazingly, all the altitudes must also meet at a single point.

Can you give the proof?

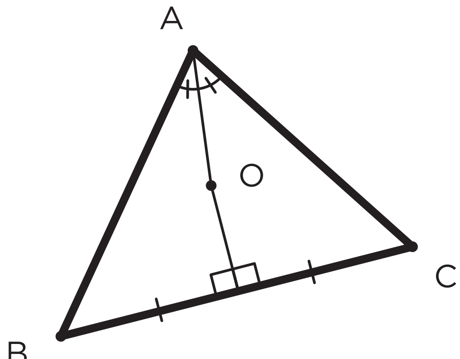


In other words, prove that if
AD and BE are altitudes meeting at a point O,
then the line CO meets AB at a right angle
(that is, is the other altitude).

Hint: This is a classic "Maze" proof. To start, can you find pairs of right triangles that share an angle? Chasing this around, you can pair up all six angles at the vertices of the triangle. Finally, how do you show the angle marked X is a right angle?

A “proof” that every triangle is equilateral.

It is remarkably difficult to spot the error! If you make a careful drawing by hand or in Geometer’s Sketchpad, you will be able to discover where the flaw is.
Hint: all the triangles that we claim are congruent really are — that’s not the problem.

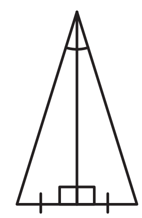


Obviously an equilateral triangle?

Given any triangle ABC, we shall prove that AB is congruent to AC. Because we may label the vertices however we please and the proof will still “work,” this implies that all three sides are congruent and the triangle is equilateral!

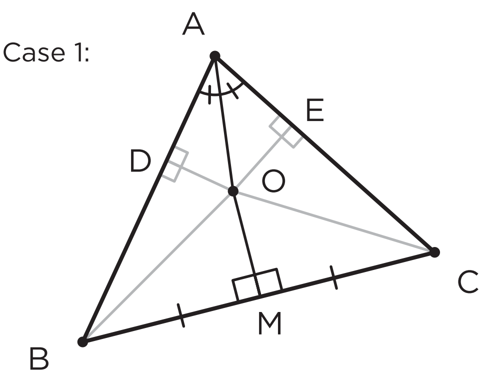
Begin by constructing the angle bisector of angle BAC and the perpendicular bisector of segment BC. Suppose for contradiction that triangle ABC is not isosceles. Then these lines are not parallel and meet at some point O.

Clearly O cannot be on segments AB or AC (since O lies on the bisector of the angle between them). And if O lies on segment BC, then the angle bisector coincides with the perpendicular bisector of BC, the triangle is isosceles (AAS), and we have proven that AB is congruent to AC as promised,



We still have two cases to consider: The point O is inside the triangle ABC or the point O is outside the triangle ABC.

Case 1:



Let M be the midpoint of BC.
From O, drop the perpendiculars to sides AB and AC, to points D and E, and draw segments BO and CO.

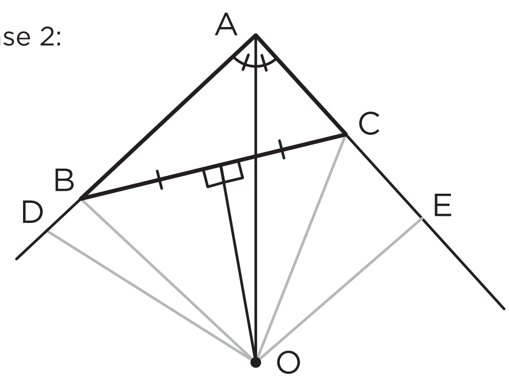
Now, triangles BMO and CMO are congruent, by SAS, so segments BO and CO are congruent.

Triangles DOA and EOA are congruent by AAS, so segments DO and EO are congruent, as are segments AD and AE.

Triangles DOB and EOC are both right triangles, and have congruent hypotenuses and a congruent leg. By the Pythagorean theorem the other legs, BD and CE, are congruent as well, and so by SSS, DOB and EOC are congruent triangles.

Segment AB is equal to AD and DB. Segment AC is equal to AE and EC. Since segment AD is congruent to segment AE, and segment DB is congruent to segment EC, we have proved that segment AB is congruent to AC.

Case 2:



Let M be the midpoint of BC.
From O, drop the perpendiculars to lines AB and AC, to points D and E, and draw segments BO and CO.

Now, triangles BMO and CMO are congruent, by SAS, so segments BO and CO are congruent.

Triangles DOA and EOA are congruent by AAS, so segments DO and EO are congruent, as are segments AD and AE.

Triangles DOB and EOC are both right triangles, and have congruent hypotenuses and a congruent leg. By the Pythagorean theorem the other legs, BD and CE, are congruent as well, and so by SSS, DOB and EOC are congruent triangles.

Segment AD is equal to AB and BD. Segment AE is equal to AC and CE. Since segment AD is congruent to segment AE, and segment DB is congruent to segment EC, we have proved that segment AB is congruent to AC.